On the Nature of Generating Functions for Singular Walks in the Quarter Plane

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Inria informatics mathematics

A Quarter Century for a Quarter Plane

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Lattice walks with small steps in the quarter plane

▷ Walks in $\mathbb{N}^2$ starting at $(0,0)$ and using steps in a fixed subset $\mathcal{S}$ of

$$\{\nwarrow, \uparrow, \nearrow, \rightarrow, \searrow, \downarrow, \swarrow, \leftarrow\}.$$

▷ Counting sequence $q_{\mathcal{S}}(i,j;n)$: number of walks of length n ending at (i,j).

▷ Full generating function:

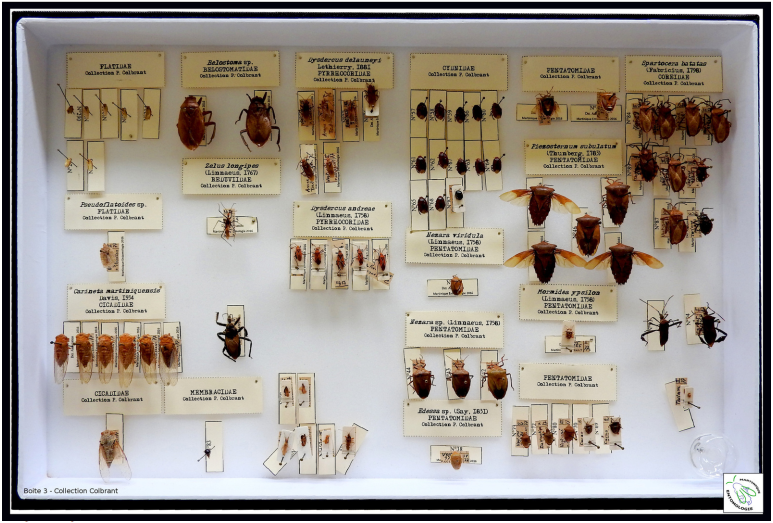
$$Q_{\mathcal{S}}(x,y;t) = \sum_{i,j,n=0}^{\infty} q_{\mathcal{S}}(i,j;n) x^i y^j t^n \in \mathbb{Z}[[x,y,t]].$$

▷ Length generating function:

$$Q_{\mathcal{S}}(1,1;t) = \sum_{i,j,n=0}^{\infty} q_{\mathcal{S}}(i,j;n) t^n \in \mathbb{Z}[[t]].$$

79 models $\mathcal{S}$

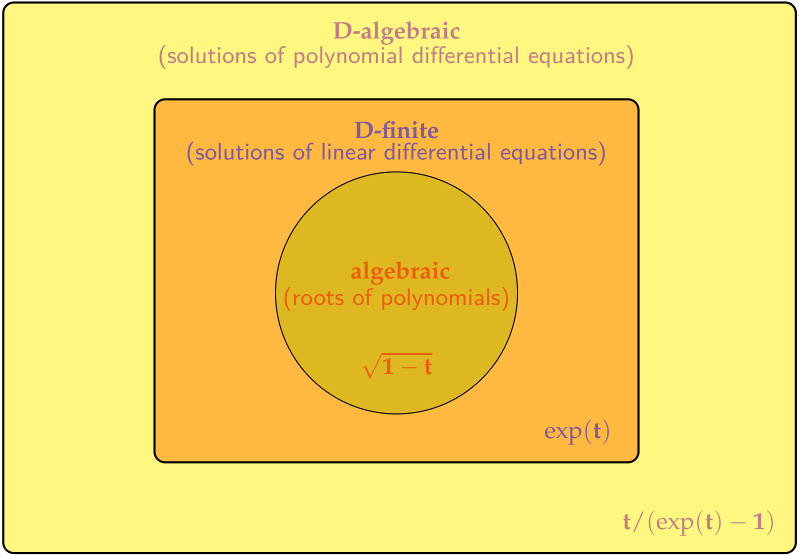
Main task: classify generating functions $Q_{\mathcal{S}}(1,1;t)$ and $Q_{\mathcal{S}}(x,y;t)$



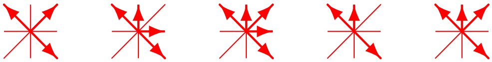
Singular

Classification of functions

D-transcendental $\Gamma(t) = \int_0^\infty x^{t-1} e^{-x}\, dx$



This talk: focus on singular models



$\mathcal{A}$ $\mathcal{B}$ $\mathcal{C}$ $\mathcal{D}$ $\mathcal{E}$

Also called *genus-zero models*, for the following reason:

In all the five cases, the *kernel polynomial*

$$K_{\mathcal{S}}(x,y,t) := xy(1 - t \cdot \chi_{\mathcal{S}}(x,y))$$

is irreducible in $\mathbb{C}(t)[x,y]$ and defines an algebraic curve of *genus zero* for generic t, where $\chi_{\mathcal{S}}$ is the characteristic (generating/inventory) polynomial

$$\chi_{\mathcal{S}}(x,y) := \sum_{(i,j)\in\mathcal{S}} x^i y^j.$$

▷ [Baker 1893] $\text{genus}(K) \leq \#\Big\{$lattice points in interior of $\mathsf{NewtonPolygon}(K)\Big\}$

Task: solve

$$K_{\mathcal{S}}(x,y,t)Q_{\mathcal{S}}(x,y;t) = xy - tx^2 Q_{\mathcal{S}}(x,0;t) - ty^2 Q_{\mathcal{S}}(0,y;t)$$

where

$$K_{\mathcal{S}}(x,y,t) := xy - t \cdot \sum_{(i,j)\in\mathcal{S}} x^{i+1} y^{j+1}$$

Theorem [Mishna, Rechnitzer, 2009], [Melczer, Mishna, 2013]

- $Q_{\mathcal{S}}(x, y; t)$ *is not D-finite* w.r.t. t;
- $Q_{\mathcal{S}}(1, 1; t)$, $Q_{\mathcal{S}}(x, 0; t)$, $Q_{\mathcal{S}}(0, y; t)$, $Q_{\mathcal{S}}(1, 0; t)$, $Q_{\mathcal{S}}(0, 1; t)$ *are not D-finite* w.r.t. t.

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Theorem [Dreyfus, Hardouin, Roques, Singer, 2020]

If $t_0 \in (0, 1/\#\mathcal{S}) \setminus \overline{\mathbb{Q}}$, then $Q_{\mathcal{S}}(x,y;t_0)$ *is D-transcendental* w.r.t. x and y.

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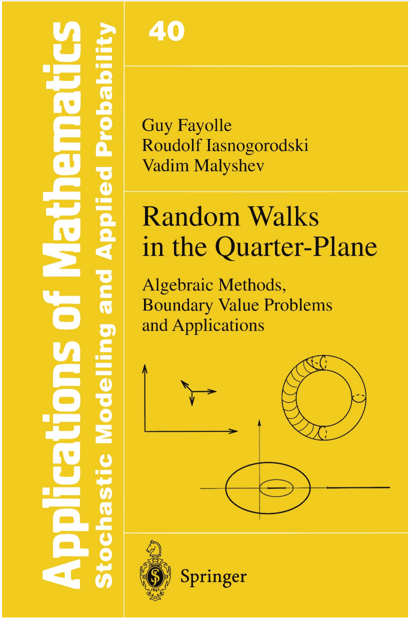
$Q_{\mathcal{S}}(x, y; t)$ *is D-transcendental* w.r.t. t.

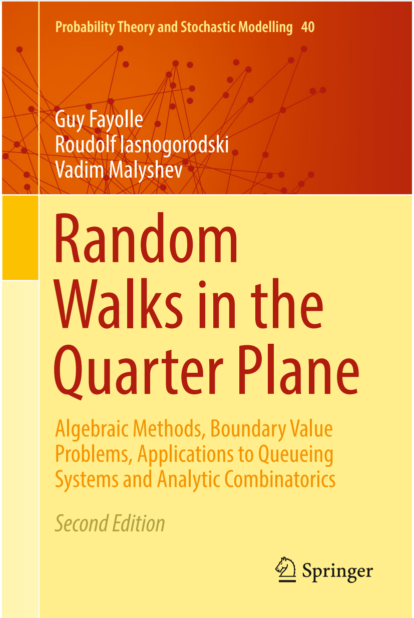
▷ Open: is the length generating function $Q_{\mathcal{S}}(1, 1; t)$ *D-transcendental w.r.t. t*?

Approach in [Dreyfus, Hardouin, Roques, Singer, 2020]

1. From the kernel equation, get functional equations satisfied by the sections $Q(x,0)$ and $Q(0,y)$ on the kernel curve $K(x,y) = 0$
2. Exhibit a rational parametrization of the kernel curve, with a special symmetry property
3. Use the parametrization to deduce a q-difference equation
4. Use Galois theory for q-difference equations (Ishizaki-Ogawara)
5. Conclude D-transcendence by reasoning on poles of a rational function

. . . based on an influential little yellow book





New results

Obtained jointly with Lucia Di Vizio and Kilian Raschel

Theorem (radius of convergence 1/2 w.r.t. t)

The series $Q_{\mathcal{S}}(x,y,t)$ converges for $|x|, |y| < 1$ and $|t| < 1/2$. Moreover,

$$Q_{\mathcal{S}}(x,y,\tfrac{1}{2}) = \sum_{i,j\geq 0} \left(\sum_{n\geq 0} \frac{\#_{\mathcal{S}}\{(0,0) \xrightarrow{n} (i,j)\}}{2^n} \right) x^i y^j \in \mathbb{Q}[[x,y]]$$

is well defined (i.e., all its coefficients are finite and define rational numbers).

▷ Proof inspired by [Mishna, Rechnitzer, 2009]

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▷ E.g.,

- $Q_{\mathcal{A}}(x,0,\pm\frac{1}{2}) = 1 + \frac{1}{2}x^2 + x^4 + \frac{17}{4}x^6 + 31x^8 + \frac{691}{2}x^{10} + 5461x^{12} + \cdots$
- $Q_{\mathcal{B}}(x,0,\frac{1}{2}) = 1 + x + 2x^2 + 7x^3 + 38x^4 + 295x^5 + 3098x^6 + 42271x^7 + \cdots$
- $Q_{\mathcal{B}}(x,0,-\frac{1}{2}) = 1 - \frac{1}{3}x + \frac{1}{15}x^3 - \frac{1}{21}x^5 + \frac{1}{15}x^7 - \frac{5}{33}x^9 + \frac{691}{1365}x^{11} + \cdots$
- $Q_{\mathcal{C}}(x,0,\frac{1}{2}) = 1 + x + \frac{5}{2}x^2 + 10x^3 + \frac{119}{2}x^4 + 493x^5 + \frac{21739}{4}x^6 + + \cdots$
- $Q_{\mathcal{C}}(x,0,-\frac{1}{2}) = 1 - \frac{1}{3}x + \frac{1}{2}x^2 - \frac{2}{5}x^3 + \frac{5}{6}x^4 - x^5 + \frac{11}{4}x^6 - \frac{197}{45}x^7 + \cdots$
- $Q_{\mathcal{D}}(x,0,\pm\frac{1}{2}) = 1 + \frac{1}{3}x + \frac{1}{3}x^2 + \frac{3}{5}x^3 + \frac{5}{3}x^4 + \frac{691}{105}x^5 + 35x^6 + \frac{3617}{15}x^7 + \cdots$
- $Q_{\mathcal{E}}(x,0,\pm\frac{1}{2}) = 1 + \frac{1}{3}x + \frac{5}{6}x^2 + \frac{28}{15}x^3 + \frac{41}{6}x^4 + \frac{3422}{105}x^5 + \frac{12137}{60}x^6 + \cdots$

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▷ Numerical coincidence?!

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▷ 12th Bernoulli number: $B_{12} = -691/(2\cdot 13\cdot 105)$

Intermezzo: Bernoulli numbers [Bernoulli, 1713]

$(B_n)_{n\geq 0} = \left(1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, \dots\right)$, the sequence of Bernoulli numbers

Summae Potestatum

$\int n = \frac{1}{2}nn + \frac{1}{2}n$
$\int nn = \frac{1}{3}n^3 + \frac{1}{2}nn + \frac{1}{6}n$
$\int n^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}nn$
$\int n^4 = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n$
$\int n^5 = \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}nn$
$\int n^6 = \frac{1}{7}n^7 + \frac{1}{2}n^6 + \frac{1}{2}n^5 - \frac{1}{6}n^3 + \frac{1}{42}n$
$\int n^7 = \frac{1}{8}n^8 + \frac{1}{2}n^7 + \frac{7}{12}n^6 - \frac{7}{24}n^4 + \frac{1}{12}nn$
$\int n^8 = \frac{1}{9}n^9 + \frac{1}{2}n^8 + \frac{2}{3}n^7 - \frac{7}{15}n^5 + \frac{2}{9}n^3 - \frac{1}{30}n$
$\int n^9 = \frac{1}{10}n^{10} + \frac{1}{2}n^9 + \frac{3}{4}n^8 - \frac{7}{10}n^6 + \frac{1}{2}n^4 - \frac{1}{12}nn$
$\int n^{10} = \frac{1}{11}n^{11} + \frac{1}{2}n^{10} + \frac{5}{6}n^9 - 1n^7 + 1n^5 - \frac{1}{2}n^3 + \frac{5}{66}n$

Quin imò qui legem progressionis inibi attentuis ensperexit, eundem etiam continuare poterit absque his ratiociniorum ambabimus : Sumtâ enim c pro potestatis cujuslibet exponente, fit summa omnium n^c seu

$$\int n^c = \frac{1}{c+1}n^{c+1} + \frac{1}{2}n^c + \frac{c}{2}An^{c-1} + \frac{c\cdot c-1\cdot c-2}{2\cdot 3\cdot 4}Bn^{c-3}$$
$$+\frac{c\cdot c-1\cdot c-2\cdot c-3\cdot c-4}{2\cdot 3\cdot 4\cdot 5\cdot 6}Cn^{c-5}$$
$$+\frac{c\cdot c-1\cdot c-2\cdot c-3\cdot c-4\cdot c-5\cdot c-6}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}Dn^{c-7}\dots$$ & ita deinceps,

exponentem potestatis ipsius n continué minuendo binario, quosque perveniatur ad n vel nn. Literae capitales A, B, C, D & c. ordine denotant coëfficientes ultimorum terminorum pro $\int nn$, $\int n^4$, $\int n^6$, $\int n^8$, & c. nempe

$$A = \frac{1}{6}, B = -\frac{1}{30}, C = \frac{1}{42}, D = -\frac{1}{30}.$$

$$\sum_{k=0}^{n} k^{12} = \frac{1}{13}n^{13} + \frac{1}{2}n^{12} + n^{11} - \frac{11}{6}n^9 + \frac{22}{7}n^7 - \frac{33}{10}n^5 + \frac{5}{3}n^3 - \frac{691}{2730}n$$

Intermezzo: Bernoulli numbers [Bernoulli, 1713]

$(B_n)_{n\geq 0} = \left(1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, \ldots\right)$, the sequence of Bernoulli numbers

▷ Explicit formula [Kronecker, 1883]

$$B_n = \sum_{k=0}^{n} (-1)^k \binom{n+1}{k+1} \frac{0^n + 1^n + \cdots + k^n}{k+1}$$

▷ Its *exponential generating function* (EGF)

$$\sum_{n\geq 0} B_n \frac{x^n}{n!} = \frac{x}{e^x - 1}$$

is D-algebraic

▷ Its *ordinary generating function* (OGF)

$$B(x) = \sum_{n\geq 0} B_n x^n$$

satisfies

$$B\left(\frac{x}{1+x}\right) = (x+1)B(x) - \frac{x}{1+x}$$

so it is *strongly D-transcendental* [B., Di Vizio, Raschel, 2024]

(IIa) Exceptional points $t = \pm\frac{1}{2}$: link with Bernoulli numbers

Theorem A

$$Q_{\mathcal{A}}(x,0;\pm 1/2) = 2\sum_{n=0}^{\infty}(2^{2n+2}-1)\frac{(-1)^n}{n+1}B_{2n+2}x^{2n}.$$

Theorem B+

$$Q_{\mathcal{B}}(x,0;1/2) = 2\cdot\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n\sum_{k=0}^{n+1}\binom{n+1}{k}\left(2^{n+k+2}-1\right)B_{n+k+2}.$$

Theorem B−

$$Q_{\mathcal{B}}(x,0;-1/2) = -2\cdot\sum_{n=0}^{\infty}B_{n+1}x^n.$$

Theorem D

$$Q_{\mathcal{D}}(x,0;\pm 1/2) = 2\cdot\sum_{n=0}^{\infty}(2n+3)B_{2n+2}(-x)^n.$$

▷ Direct proof of *D-transcendence of $Q_{\mathcal{S}}(x,0;t)$* w.r.t. x for $\mathcal{S}\in\{\mathcal{A},\mathcal{B},\mathcal{D}\}$

(IIb) Exceptional points $t = \pm\frac{1}{2}$: D-algebraicity vs. D-transcendence

Theorem A

Let $Q_{\mathcal{A}}(x, 0; \pm 1/2) = \sum_{n \geq 0} a_n x^n$. Then $F_{\mathcal{A}}(x) := \sum_{n \geq 0} \frac{a_n}{(n+1)!} x^n$ is *D-algebraic*:

$$x^2 F_A(x)^2 - 4x F_A'(x) - 4F_A(x) + 4 = 0.$$

Theorem B−

Let $Q_{\mathcal{B}}(x, 0; -1/2) = \sum_{n \geq 0} b_n x^n$. Then $F_{\mathcal{B}}(x) := \sum_{n \geq 0} \frac{b_n}{(n+1)!} x^n$ is *D-algebraic*:

$$x F_B(x)^2 - (2x+4) F_B'(x) - 2x F_B(x) + 4 = 0.$$

Theorem D

Let $Q_{\mathcal{D}}(x, 0; \pm 1/2) = \sum_{n \geq 0} d_n x^n$. Then $F_{\mathcal{D}}(x) := \sum_{n \geq 0} \frac{d_n}{(2n+3)!} x^n$ is *D-algebraic*:

$$x F_D(x)^2 - 4x F_D'(x) - 6F_D(x) + 1 = 0.$$

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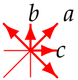
▷ Another instance of the Pak-Yeliussizov conjecture (2018)



$$K_{\mathcal{S}}\left(x, y, \frac{1}{2}\right) := xy - \frac{1}{2} \cdot \left(ax^2y^2 + bxy^2 + cx^2y + x^2 + y^2\right)$$

Theorem (uniform parametrization)

When $b + c \neq 0$, the curve $K_{\mathcal{S}}\left(x, y, \frac{1}{2}\right) = 0$ is parametrized by

$$(x(s), y(s)) := \left(\frac{-s^2}{\frac{a}{b+c}\, s^2 - c\, s + (b+c)}, \frac{-s^2}{\frac{a}{b+c}\, s^2 + b\, s + (b+c)}\right)$$

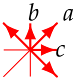
with the additional property that

$$(\tilde{x}(s), y(s)) := \left(x\left(\frac{s}{s+1}\right), y(s)\right)$$

is another parametrization of the same curve.

▷ Similar (simpler) parametrization if $b + c = 0$

(IIIb) Difference equation at the exceptional point $t = \frac{1}{2}$



Theorem (uniform difference equation)

When $b + c \neq 0$, the power series $Q_{\mathcal{S}}\left(x, 0, \frac{1}{2}\right) =$

$$1 + \left(\frac{b}{3} + \frac{2c}{3}\right) x + \left(\frac{1}{3} b^2 + bc + \frac{2}{3} c^2 + \frac{1}{2} a\right) x^2 + \left(\frac{3}{5} b^3 + \frac{12}{5} b^2 c + \frac{44}{15} c^2 b + \frac{16}{15} c^3 + \frac{19}{15} ab + \frac{26}{15} ac\right) x^3 + \cdots$$

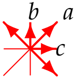
satisfies $G(s) - G\left(\frac{s}{1+s}\right) = R(s)$, where $R(s) =$

$$\frac{(b\,s + 2(b+c))\; s^5}{\left(\frac{a}{b+c}\, s^2 + b\,s + (b+c)\right)\left(\frac{a}{b+c}\, s^2 - c\,s + (b+c)\right)\left(\left(\frac{a}{b+c} + b\right) s^2 + (2b+c)\,s + (b+c)\right)}$$

and $$G(s) = \frac{1}{2} \cdot \left(\frac{-s^2}{\frac{a}{b+c}\, s^2 - c\,s + (b+c)}\right)^2 \cdot Q_{\mathcal{S}}\left(\frac{-s^2}{\frac{a}{b+c}\, s^2 - c\,s + (b+c)}, 0, \frac{1}{2}\right).$$

▷ Similar statement for $t = -1/2$

(IIIc) Strong D-transcendence for $t = \frac{1}{2}$



Theorem (Strong D-transcendence)

For $\mathcal{S} \in \{\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}\}$, the power series $Q_{\mathcal{S}}\left(x, 0, \frac{1}{2}\right) =$

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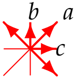

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▷ Implies the strong D-transcendence of the generating functions

- If $(a, b, c) = (1, 0, 0)$, then $Q_{\mathcal{A}}(x, 0, \frac{1}{2}) = 1 + \frac{1}{2}x^2 + x^4 + \frac{17}{4}x^6 + 31x^8 + \cdots$

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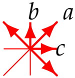

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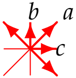

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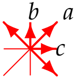
(IIIc) Strong D-transcendence for $t = \frac{1}{2}$



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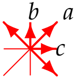
- If $(a, b, c) = (0, 1, 0)$, then $Q_{\mathcal{D}}(x, 0, \frac{1}{2}) = 1 + \frac{1}{3}x + \frac{1}{3}x^2 + \frac{3}{5}x^3 + \frac{5}{3}x^4 + \cdots$

(IIIc) Strong D-transcendence for $t = \frac{1}{2}$



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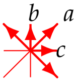
- If $(a, b, c) = (1, 1, 0)$, then $Q_{\mathcal{E}}(x, 0, \frac{1}{2}) = 1 + \frac{1}{3}x + \frac{5}{6}x^2 + \frac{28}{15}x^3 + \frac{41}{6}x^4 + \cdots$

▷ Refines [Dreyfus, Hardouin, Roques, Singer, 2020]: for $t_0 \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, the function $Q_{\mathcal{S}}(x, y; t_0)$ *is D-transcendental* w.r.t. x and y.

(IV) A new phenomenon: D-algebraic cases for $t = \frac{1}{2}$



Theorem (D-algebraic cases)

Let $b + c \neq 0$, $\ell \in \mathbb{Z} \setminus \{0, -1\}$ and

$$a = -\frac{\ell(\ell+1)(\ell b + (\ell+1)c)((\ell+1)b + \ell c)}{(2\ell+1)^2}.$$

Then the power series $Q_{\mathcal{S}}\left(x, 0, \frac{1}{2}\right)$ is rational.

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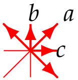

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▷ If $(a, b, c, \ell) = (-24, 2, 5, 1)$, then $Q_{\mathcal{S}}\left(x, 0, \frac{1}{2}\right) = \frac{1}{1-4x}$

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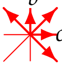

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- ▷ If $(a, b, c, \ell) = (-12, 1, 1, 3)$, then $Q_{\mathcal{S}}\left(x, 0, \frac{1}{2}\right) = \frac{-5/2}{3x-1} + \frac{5/2}{5x-1} + \frac{-1}{6x-1}$

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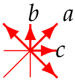

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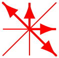
- ▷ More generally, if $(a, b, c) = (-\ell(\ell+1), 1, 1)$, with $\ell \in \mathbb{N} \setminus \{0\}$ then

$$Q_{\mathcal{S}}\left(x, 0, \frac{1}{2}\right) = 1 + x + \left(-\frac{\ell^2+\ell}{2} + 2\right)x^2 + \left(-3\ell^2 - 3\ell + 7\right)x^3 + \left(\ell^4 + 2\ell^3 - \frac{39}{2}\ell^2 - \frac{41}{2}\ell + 38\right)x^4 + \cdots$$

is in $\mathbb{Q}(x)$, of degree ℓ, of the form $\sum_{m=1}^{\ell} \frac{c_m}{(m\ell - \binom{m}{2})x - 1}$

The model $\mathcal{B} = \{N, E, NW, SE\}$



(V) Explicit expression for model $\mathcal{B} = \{N, E, NW, SE\}$

Theorem (model $\mathcal{B}$)

Let $(M_n)_{n\geq 0} = (1, 2, 8, 56, 608, \ldots)$ be *median Genocchi numbers* (A005439).

$$Q_{\mathcal{B}}\left(x, 0; \frac{1}{2}\right) = 1 + x + 2\,x^2 + 7\,x^3 + 38\,x^4 + 295\,x^5 + 3098\,x^6 + \cdots$$

is strongly D-transcendental, equal to $\displaystyle\sum_{n\geq 0} \frac{M_n}{2^n}\,x^n$.

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▷ Kernel curve $K_{\mathcal{B}}(x, y, \frac{1}{2}) = xy - \frac{1}{2}\left(x^2y + xy^2 + x^2 + y^2\right)$ parametrized by

$$x(s) = -\frac{2s^2}{s+1} \quad \text{and} \quad y(s) = -\frac{2s^2}{1-s}.$$

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▷ Kernel equation implies that $f(s) := Q_{\mathcal{B}}(s, 0; \frac{1}{2})$ satisfies the equation

$$\frac{1}{(s+1)^2}\, f\left(-\frac{2s^2}{s+1}\right) + \frac{1}{(1-s)^2}\, f\left(-\frac{2s^2}{1-s}\right) = \frac{2}{1-s^2}. \qquad (\star)$$

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▷ [Dumont, Zeng, 1994]: $f(s) := \displaystyle\sum_{n \geq 0} M_n\, (s/2)^n$ also satisfies $(\star)$

▷ [Barsky, Dumont, 1981]: coefficients are integers! (*Dellac sequence* A000366)

(VIa) D-algebraicity for model $\mathcal{B} = \{N, E, NW, SE\}$

Theorem (D-algebraicity for $\mathcal{B}$)

Let $C(t) = \frac{1-\sqrt{1-4t^2}}{2t} = \sum_{n\geq 0} \mathrm{Cat}_n t^{2n+1} = t + t^3 + 2t^5 + 5t^7 + 14t^9 + \cdots$, and let $\theta_2(q)$ and $\theta_3(q)$ be the Jacobi theta series

$$\theta_2(q) = 1 + 2\left(q + q^4 + q^9 + q^{16} + q^{25} + \cdots\right),$$

$$\theta_3(q) = 2\left(q^{1/4} + q^{9/4} + q^{25/4} + q^{49/4} + \cdots\right).$$

Then,

$$Q_{\mathcal{B}}(-1, 0; t) = \frac{1-4t^2}{t^3} \cdot \frac{\theta_2^4(C(t)) + \theta_3^4(C(t)) - 1}{24} - \frac{t+1}{t^2}.$$

$(= 1 - t + t^3 - t^4 + t^5 - t^6 + t^8 - 3t^9 + 6t^{10} - 9t^{11} + 14t^{12} - 17t^{13} + 20t^{14} - 17t^{15} + 2t^{16} + 30t^{17} - 110t^{18} + \cdots)$

In particular, $Q_{\mathcal{B}}(-1, 0; t)$ and $Q_{\mathcal{B}}(-1, -1; t)$ are D-algebraic.

▷ Proof combines the iterated kernel method [Mishna, Rechnitzer, 2009] and an identity due to Jacobi-Ramanujan.

(VIb) D-algebraicity for model $\mathcal{B} = \{N, E, NW, SE\}$

Theorem (model $\mathcal{B}$)

Let $T = q/(1+q^2)$ and $X = (1-q)^2/(2q)$. Then, $Q_{\mathcal{B}}(X, 0; T)$ is D-algebraic (in q).

Moreover, the following identity holds:

$$1 + T \cdot X \cdot Q_{\mathcal{B}}(X, 0; T) = (X+2)/2 \cdot (1 - \theta_4(q)^4),$$

where $\theta_4(q)$ is the Jacobi theta series

$$\theta_4(q) = 1 - 2q + 2q^4 - 2q^9 + 2q^{16} - 2q^{25} + \cdots .$$

▷ Similar proof.

Questions on our todo list

▷ *Bernoulli numbers* pop up in models $\mathcal{S} \in \{\mathcal{A}, \mathcal{B}, \mathcal{D}\}$ for $t = \pm\frac{1}{2}$.

⟶ Is there a *combinatorial explanation*?

⟶ Are there similar results for models $\mathcal{C}, \mathcal{E}$?

⟶ Is there a lifting to some *q-Bernoulli polynomials* for all $t = q/(q^2+1)$?

⟶ Are there systematically some D-algebraic EGFs behind the scenes?

▷ Characterize all weighted models with *D-algebraic* $Q(x, 0, \pm 1/2)$

▷ Coeffs. of D-transcendental series *may satisfy* non-linear recurrences!

⟶ Is the concept of D-transcendence pertinent to combinatorics?

▷ The Pak-Yeliussizov problem: Characterize the power series $F(x) = \sum_n a_n x^n$ such that *both F and* $F \odot \exp(x) = \sum_n a_n \frac{x^n}{n!}$ are *D-algebraic*.

Thanks for your attention!

Long Live Guy and the Little Yellow Book!