Counting lattice walks by winding angle

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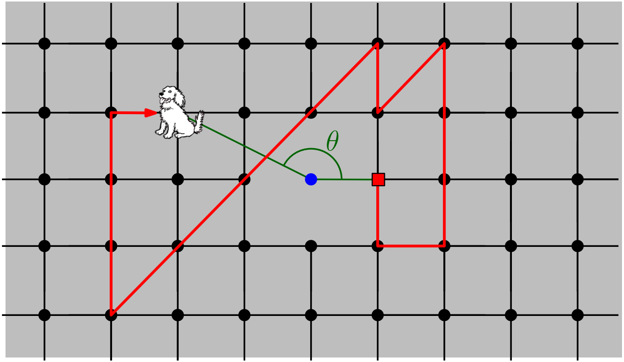
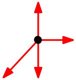
Lattice walks by winding angle

TALK OUTLINE

- **Part 1:** Introduction to winding
- **Part 2:** Functional equations
- **Part 3:** Solution to functional equations
  - Conversion to analytic functional equation à la petit livre jaune
  - Solution to analytic functional equation
- **Part 4:** Some nice special cases

Part 1: Introduction to winding





[Spitzer, 1958]: The winding θ_t of two-dimensional Brownian motion at time t behaves like

$$\lim_{t\to\infty} p\left(x = \frac{2\theta_t}{\log(t)}\right) = \frac{1}{\pi}\frac{1}{1+x^2}.$$

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Not realistic: dog can't go indefinitely close to legs

History of winding

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[Pitman, Yor, 1986]: For the same model, the part $\tilde{\theta}_t$ of the winding that happens outside some disk around • behaves like

$$\lim_{t\to\infty} p\left(x = \frac{2\tilde{\theta}_t}{\log(t)}\right) = \frac{1}{e^{\frac{\pi x}{2}} + e^{-\frac{\pi x}{2}}}.$$

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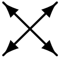
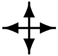
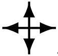
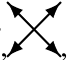
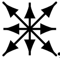
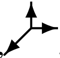
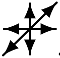
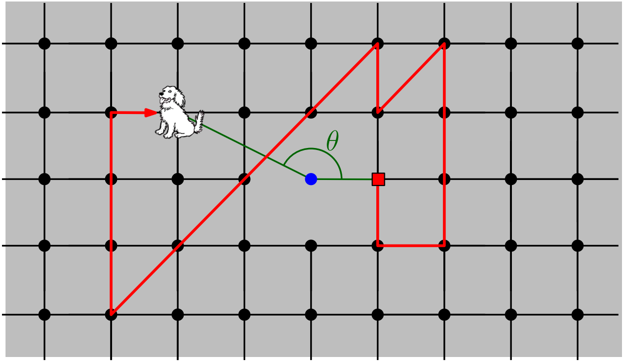
Claim: Same behaviour for our models

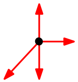
[Timothy Budd, 2017]: Solved for step-sets  starting and ending on an axis (equivalently  starting and ending on a diagonal)

[E.P., 2020 (FPSAC)]: Solved for , , , , .

This work: Solution for any step set $S \in \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$.

The model: Choose a step set $S \in \{-1,0,1\}^2 \setminus \{(0,0)\}$.
Problem: Determine the number $q_{i,j,n,\theta}$ of paths in $\mathbb{Z}^2$ avoiding $(0,0)$ with steps in S from $(1,0)$ to (i,j) of length n and winding angle θ.





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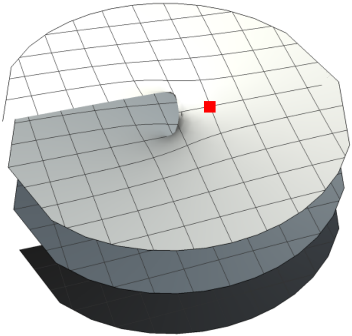
Equivalently: count walks starting at ■ by length and end point.



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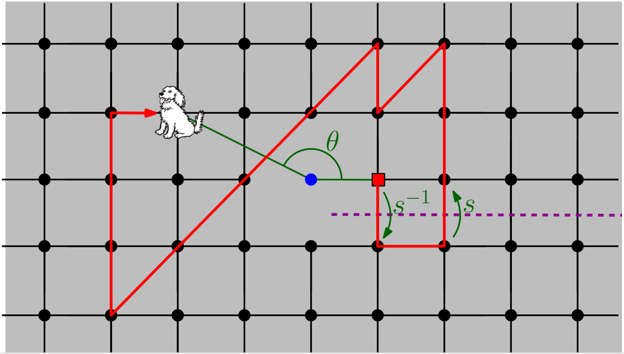
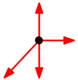
Equivalently: Determine the generating function

$$\mathsf{W}(x,y;t,s) := \sum_{i,j,n,\theta} q_{i,j,n,\theta} x^i y^j t^n s^{\lfloor \frac{\theta}{2\pi} \rfloor}.$$





Walks with small steps by winding angle

The model: Choose a step set $S \in \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$.
Problem: Determine the number $q_{i,j,n,\theta}$ of paths in $\mathbb{Z}^2$ avoiding $(0, 0)$ with steps in S from $(1, 0)$ to (i, j) of length n and winding angle θ.
Equivalently: Determine the generating function

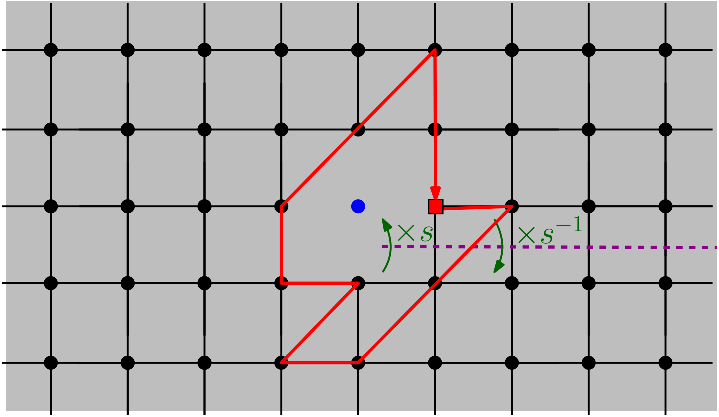
$$\mathsf{W}(x, y; t, s) := \sum_{i,j,n,\theta} q_{i,j,n,\theta} x^i y^j t^n s^{\left\lfloor \frac{\theta}{2\pi} \right\rfloor}.$$

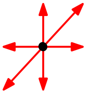
Equivalent because: θ can be recovered from monomial using

$$\theta = 2\pi \left\lfloor \frac{\theta}{2\pi} \right\rfloor + Arg(x + iy)$$

where we define $Arg(z) \in [0, 2\pi)$.

Preview: Double Kreweras excursions





Preview: Double Kreweras excursions

Define

$$T(u,q) = \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2} (u^{n+1} - u^{-n}), .$$

Let $q := e^{2i\gamma} = t + t^2 + 7t^3 + 23t^4 + 122t^5 + 554t^6 + \cdots$ be the unique series satisfying

$$q \frac{T(q,q^6)^3}{T(q^3,q^6)^3} = \frac{t}{1+2t}.$$

The generating function $\mathsf{E}(t,s)$ for double kreweras excursions $(1,0) \to (1,0)$ is given by

$$\mathsf{E}(t,s) = \frac{\left(T\left(q^3,q^6\right)T\left(q^2 s,q^6\right) - 2\,T\left(q^3 s,q^6\right)T\left(q^2,q^6\right) + T\left(q^3,q^6\right)T\left(\frac{q^2}{s},q^6\right)\right)}{tq(1-s^{-1})T\left(q,q^6\right)T\left(s,q^6\right)}$$
$$= 1 + 5t^2 + 8t^3 + 62t^4 + 216t^5 + (s^{-1} + 1199 + s)t^6 + \cdots$$

Preview: Double Kreweras excursions (using ϑ)

Define the Jacobi theta function

$$\vartheta(z,\tau) := \sum_{n=0}^{\infty}(-1)^n e^{i\pi\tau(n+\frac{1}{2})^2}\left(e^{(2n+1)iz}-e^{-(2n+1)iz}\right)$$
$$= -ie^{\frac{i\pi\tau}{4}}e^{-iz}T(e^{2iz},e^{2i\pi\tau}).$$

Let $\gamma = \frac{\pi\tau}{6}$ and τ be determined by

$$e^{-4i\gamma}\frac{\vartheta(\gamma,\tau)^3}{\vartheta(3\gamma,\tau)^3} = \frac{t}{1+2t},$$

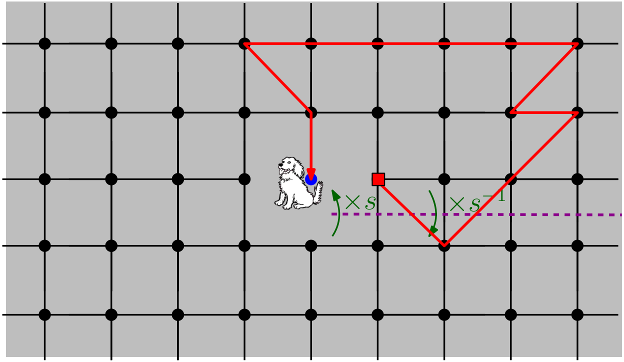
The generating function $\mathsf{E}(t,s)$ for double kreweras excursions $(1,0)\to(1,0)$ is given by

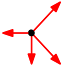
$$\mathsf{E}(t,e^{2i\kappa}) = -\frac{\left(2\vartheta\left(2\gamma\right)\vartheta\left(3\gamma+\kappa\right)-\vartheta\left(3\gamma\right)\vartheta\left(2\gamma+\kappa\right)-e^{-2i\kappa}\vartheta\left(3\gamma\right)\vartheta\left(2\gamma-\kappa\right)\right)e^{2i\gamma}}{t\left(1-e^{-2i\kappa}\right)\vartheta\left(\gamma\right)\vartheta\left(\kappa\right)}$$
$$= 1+5t^2+8t^3+62t^4+216t^5+(s^{-1}+1199+s)t^6+\cdots$$

Part 2: Functional equations for walks by winding number

Some refined walk types and generating functions

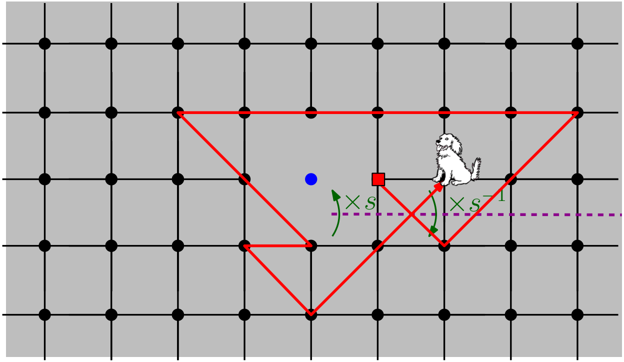
- Terminating walk, $\mathsf{T}(t, s)$: walk ending at $(0, 0)$

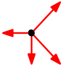




Some refined walk types and generating functions

- Terminating walk, $\mathsf{T}(t, s)$: walk ending at $(0, 0)$
- Rising walk, $\mathsf{R}(x; t, s)$: final step moves up through dotted line





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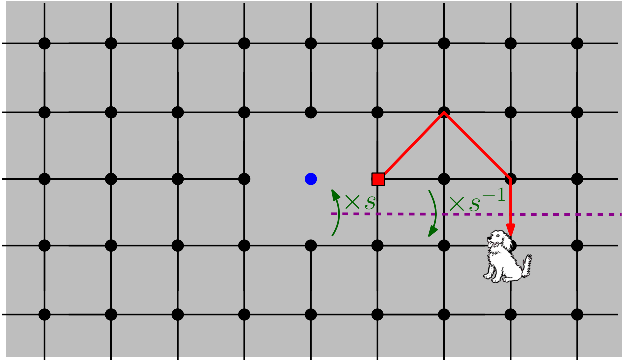
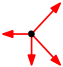
- Terminating walk, $\mathsf{T}(t,s)$: walk ending at $(0,0)$
- Rising walk, $\mathsf{R}(x;t,s)$: final step moves up through dotted line
- Sinking walk, $\frac{1}{y}\mathsf{S}(x;t,s)$: final step moves down through dotted line





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No winding factor:

$$\begin{array}{ccccc} (\textbf{Walk} + \textbf{step}) & \text{or } \textbf{empty walk} = & \textbf{walk} & \text{or } \textbf{Terminating walk} \\ \mathsf{W}(x,y;t,1)tP(x,y) + & x \quad = & \mathsf{W}(x,y;t,1) + & \mathsf{T}(t,s). \end{array}$$

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With winding factor correction:

$$\mathsf{W}(x,y)tP(x,y) + x = \mathsf{W}(x,y) + \mathsf{T} + \left(\frac{1}{s} - 1\right)\mathsf{R}(x) + (s-1)\frac{1}{y}\mathsf{S}(x)$$

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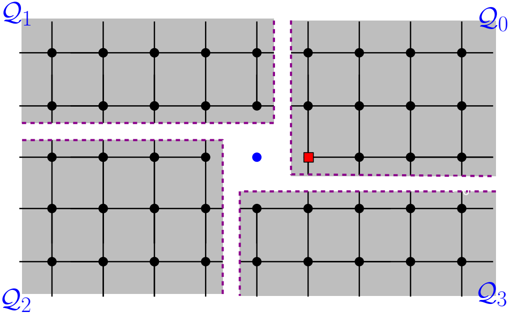
Uniquely defines series (along with relation coming from definition of R and S) but hard to analyse directly

Before analysis: Refine functional equation

Refinement: Write

$$\mathsf{W}(x,y;t,s) = \mathsf{Q}_0(x,y;t,s)+\mathsf{Q}_1(x,y;t,s)+\mathsf{Q}_2(x,y;t,s)+\mathsf{Q}_3(x,y;t,s),$$

where each Q_j counts walks ending in a specific quadrant $\mathcal{Q}_j$.



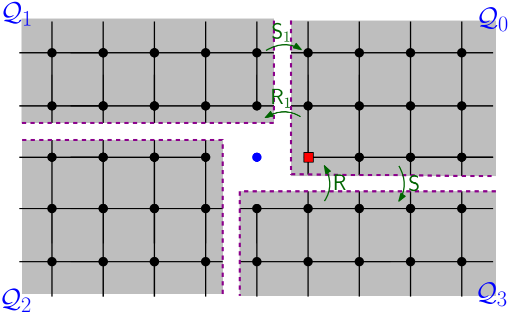
Refinement: Write

$$W(x, y; t, s) = Q_0(x, y; t, s) + Q_1(x, y; t, s) + Q_2(x, y; t, s) + Q_3(x, y; t, s),$$

where each Q_j counts walks ending in a specific quadrant $\mathcal{Q}_j$.

Equation for $Q_0(x, y; t, s)$:

$$Q_0(x, y)tP(x, y) + x = Q_0(x, y) + T_0 - R(x) + sy^{-1}S(x) + R_1(y) - xS_1(y)$$



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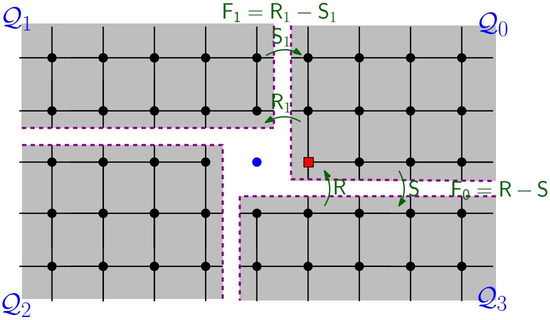
Equation for $\mathsf{Q}_0(x,y;t,s)$:

$$\mathsf{Q}_0(x,y)tP(x,y) + x = \mathsf{Q}_0(x,y) + \mathsf{T}_0 - \mathsf{F}_0(x,y) + \mathsf{F}_1(x,y)$$



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where each Q_j counts walks ending in a specific quadrant $\mathcal{Q}_j$.

Equations defining $\mathsf{Q}_j(x,y;t,s)$ for $j=0,1,2,3$:

$$\begin{aligned}
\mathsf{Q}_0(x,y)tP(x,y)+x &= \mathsf{Q}_0(x,y)+\mathsf{T}_0-\mathsf{F}_0(x,y)+\mathsf{F}_1(x,y)\\
\mathsf{Q}_1(x,y)tP(x,y) &= \mathsf{Q}_1(x,y)+\mathsf{T}_1-\mathsf{F}_1(x,y)+\mathsf{F}_2(x,y)\\
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\mathsf{Q}_3(x,y)tP(x,y) &= \mathsf{Q}_3(x,y)+\mathsf{T}_3-\mathsf{F}_3(x,y)+\mathsf{F}_4(x,y)\\
\mathsf{F}_4(x,y) &= s^{-1}\mathsf{F}_0(x,y)
\end{aligned}$$

Refinement: Write

$$\mathsf{W}(x,y;t,s)=\mathsf{Q}_0(x,y;t,s)+\mathsf{Q}_1(x,y;t,s)+\mathsf{Q}_2(x,y;t,s)+\mathsf{Q}_3(x,y;t,s),$$

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To simplify, write $K(x,y;t)=1-tP(x,y)$.

To solve:

$$\delta_{0,j}x=K(x,y;t)\mathsf{Q}_j(x,y)+\mathsf{T}_j-\mathsf{F}_j(x,y)+\mathsf{F}_{j+1}(x,y),$$

$$\mathsf{F}_4(x,y)=s^{-1}\mathsf{F}_0(x,y).$$

Part 3: Solving functional equation via analytic functional equation

à la petit petit livre jaune

To solve:

$$\delta_{0,j}x = K(x,y;t)\mathsf{Q}_j(x,y) + \mathsf{T}_j - \mathsf{F}_j(x,y) + \mathsf{F}_{j+1}(x,y),$$
$$\mathsf{F}_4(x,y) = s^{-1}\mathsf{F}_0(x,y).$$

Solution idea: Based on methods used for quadrant walks [Fayolle, Ianogorodski, 1979], [Fayolle, Ianogorodski, Malyshev, 1999], [Raschel, 2010], similarly to [EP, 2022] for 3/4-plane walks

Solving functional equations

To solve:

$$\delta_{0,j}x = K(x,y;t)\mathsf{Q}_j(x,y) + \mathsf{T}_j - \mathsf{F}_j(x,y) + \mathsf{F}_{j+1}(x,y),$$
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Step 1: Fix $t < 1/9$ and parametrise $\overline{E}_t = \{(x,y) : K(x,y;t) = 0\}$:

$$x(\omega) = x_0 + \frac{x_1}{\wp(\omega;\omega_1,\omega_2) + x_2},$$
$$y(\omega) = y_0 + \frac{y_1}{\wp(\omega - \omega_3/2;\omega_1,\omega_2) + y_2},$$
$$\overline{E}_t = \{(x(\omega), y(\omega)) : \omega \in \mathbb{C}\}.$$

Inherited transformation properties: $x(\omega)$ and $y(\omega)$ satisfy

- $x(\omega) = x(\omega+\omega_1) = x(\omega+\omega_2) = x(-\omega)$
- $y(\omega) = y(\omega+\omega_1) = y(\omega+\omega_2) = y(\omega_3-\omega)$

To solve:

$$\delta_{0,j}x = K(x,y;t)\mathsf{Q}_j(x,y) + \mathsf{T}_j - \mathsf{F}_j(x,y) + \mathsf{F}_{j+1}(x,y),$$
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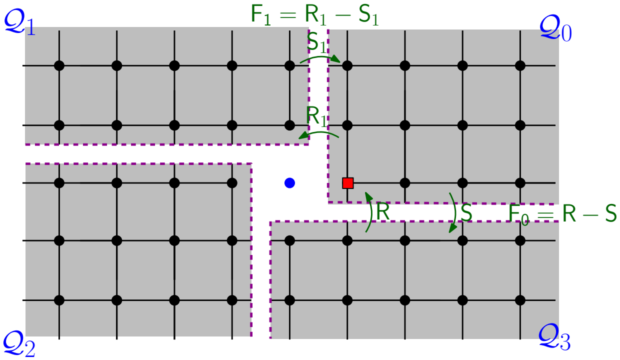
Step 2: Substitute $x \to x(\omega)$ and $y \to y(\omega)$ in functional equations

Warm up: Consider just the first equation for $t < 1/9$ and $|s| = 1$.

$$x = K(x,y)\mathsf{Q}_0(x,y) + \mathsf{T}_0 - \mathsf{F}_0(x,y) + \mathsf{F}_1(x,y)$$



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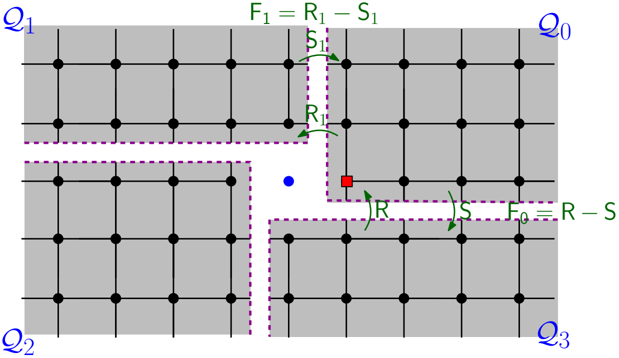
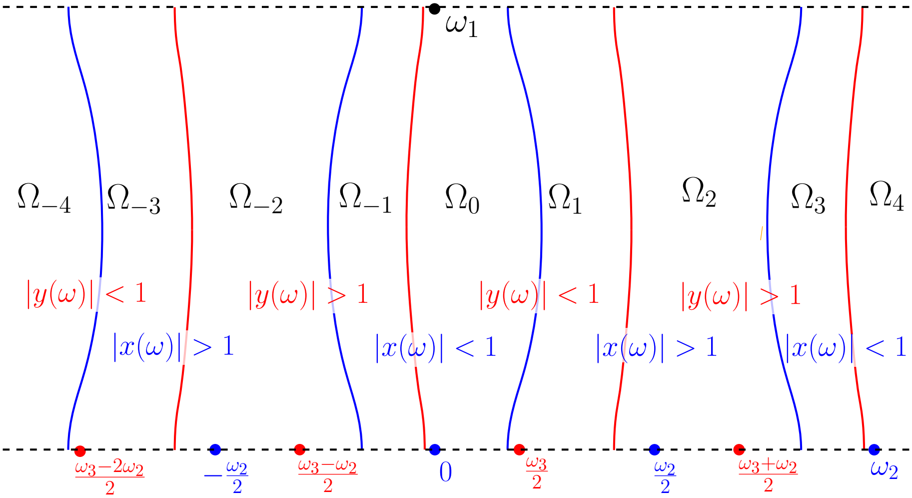
Powers of x, y positive $\Rightarrow$ Converges when $|x|, |y| < 1$.
to substitute $x \to x(\omega)$ and $y \to y(\omega)$, need to understand when $|x(\omega)| < 1$ and $|y(\omega)| < 1$.



THE FUNCTIONS $x(\omega)$ AND $y(\omega)$ FOR $\omega \in \mathbb{C}/(\omega_1\mathbb{Z})$



For $\omega \in \Omega_0$: $|x(\omega)|, |y(\omega)| < 1$, so $Q_0(x(\omega), y(\omega))$ converges.

Warm up: Consider just the first equation for $t < 1/9$ and $|s| = 1$:

$$x = K(x,y)\mathsf{Q}_0(x,y) + \mathsf{T}_0 - \mathsf{F}_0(x,y) + \mathsf{F}_1(x,y)$$

Powers of x, y positive $\Rightarrow$ Converges when $|x|, |y| < 1$.

Recall:

$\mathsf{F}_0(x,y;t,s) \in \mathbb{Z}[[x,t,s]] + \frac{1}{y}\mathbb{Z}[[x,t,s]] \Rightarrow$ Converges when $|x| < 1$

$\mathsf{F}_1(x,y;t,s) \in \mathbb{Z}[[y,t,s]] + x\mathbb{Z}[[y,t,s]] \Rightarrow$ Converges when $|y| < 1$

Define:

$$\begin{aligned} \tilde{F}_0(\omega) &= \mathsf{F}_0(x(\omega), y(\omega)), && \text{for } \omega \in \Omega_{-1} \cup \Omega_0, \\ \tilde{F}_1(\omega) &= \mathsf{F}_1(x(\omega), y(\omega)), && \text{for } \omega \in \Omega_0 \cup \Omega_1. \end{aligned}$$

Substitute $x \to x(\omega)$ and $y \to y(\omega)$ for $\omega \in \Omega_0$:

$$x(\omega) = \mathsf{T}_0 - \tilde{F}_0(\omega) + \tilde{F}_1(\omega).$$

General version: Consider all equations for $t < 1/9$ and $|s| = 1$:

$$\delta_{0,j}x = K(x,y;t)\mathsf{Q}_j(x,y) + \mathsf{T}_j - \mathsf{F}_j(x,y) + \mathsf{F}_{j+1}(x,y),$$
$$s\mathsf{F}_4(x,y) = \mathsf{F}_0(x,y).$$

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\tilde{F}_1(\omega) &= \mathsf{F}_1(x(\omega),y(\omega)), && \text{for } z \in \Omega_0 \cup \Omega_1,\\
\tilde{F}_2(\omega) &= \mathsf{F}_2(x(\omega),y(\omega)), && \text{for } z \in \Omega_1 \cup \Omega_2,\\
\tilde{F}_3(\omega) &= \mathsf{F}_3(x(\omega),y(\omega)), && \text{for } z \in \Omega_2 \cup \Omega_3,\\
\tilde{F}_4(\omega) &= \mathsf{F}_4(x(\omega),y(\omega)), && \text{for } z \in \Omega_3 \cup \Omega_4.
\end{aligned}$$

Substitute $x \to x(\omega)$ and $y \to y(\omega)$ for $\omega \in \Omega_j$ for $j = 0, 1, 2, 3$:

$$\delta_{0,j}x(\omega) = \mathsf{T}_j - \tilde{F}_j(\omega) + \tilde{F}_{j+1}(\omega).$$

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Define:

$$\tilde{F}_j(\omega) = \mathsf{F}_j(x(\omega), y(\omega)), \qquad \text{for } z \in \Omega_{j-1} \cup \Omega_j$$

Substitute $x \to x(\omega)$ and $y \to y(\omega)$ for $\omega \in \Omega_j$ for $j = 0, 1, 2, 3$:

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$$\delta_{0,j} x(\omega) = \mathsf{T}_j - \tilde{F}_j(\omega) + \tilde{F}_{j+1}(\omega).$$

Take analytic extensions to $\omega \in \mathbb{C}$ and add up equations:

To solve:

$$x(\omega) = \mathsf{T} - \tilde{F}_0(\omega) + \tilde{F}_4(\omega).$$

General version: Consider all equations for $t < 1/9$ and $|s| = 1$:

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$$\delta_{0,j}x(\omega) = \mathsf{T}_j - \tilde{F}_j(\omega) + \tilde{F}_{j+1}(\omega).$$

Final equation: for $\omega \in \Omega_0$, $\omega + \omega_2 \in \Omega_4$ and $x(\omega) = x(\omega + \omega_2)$ and $y(\omega) = y(\omega + \omega_2)$, so

$$\tilde{F}_0(\omega) = \mathsf{F}_0(x(\omega), y(\omega)) = s\mathsf{F}_4(x(\omega), y(\omega)) = s\tilde{F}_4(\omega + \omega_2).$$

Take analytic extensions to $\omega \in \mathbb{C}$ and add up equations:

To solve:

$$x(\omega) = \mathsf{T} - \tilde{F}_0(\omega) + \tilde{F}_4(\omega).$$

General version: Consider all equations for $t < 1/9$ and $|s| = 1$:

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Take analytic extensions to $\omega \in \mathbb{C}$ and add up equations:

To solve:

$$x(\omega) = \mathsf{T} - \tilde{F}_0(\omega) + s^{-1}\tilde{F}_0(\omega - \omega_2).$$

To solve:

$$x(\omega) = \mathsf{T} - \tilde{F}_0(\omega) + s^{-1}\tilde{F}_0(z\omega - \omega_2).$$

To solve:

$$x(\omega) = \mathsf{T} - \tilde{F}_0(\omega) + s^{-1}\tilde{F}_0(z\omega - \omega_2).$$

Recall: $x(\omega) = x(\omega - \omega_2)$.

To solve:

$$x(\omega) = \mathsf{T} - \tilde{F}_0(\omega) + s^{-1}\tilde{F}_0(z\omega - \omega_2).$$

Recall: $x(\omega) = x(\omega - \omega_2)$.
Simplify equation: Write $s = e^{2i\kappa}$ and define

$$A(\omega) = x(\omega) - \mathsf{T} + (1 - e^{-2i\kappa})\tilde{F}_0(\omega)$$

To solve:

$$A(\omega) = e^{-2i\kappa}A(\omega - \omega_2).$$

Other information $A(\omega) - x(\omega)$ has no poles in $\Omega_{-4}, \Omega_{-3}, \ldots, \Omega_0$.

To solve:

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To solve:

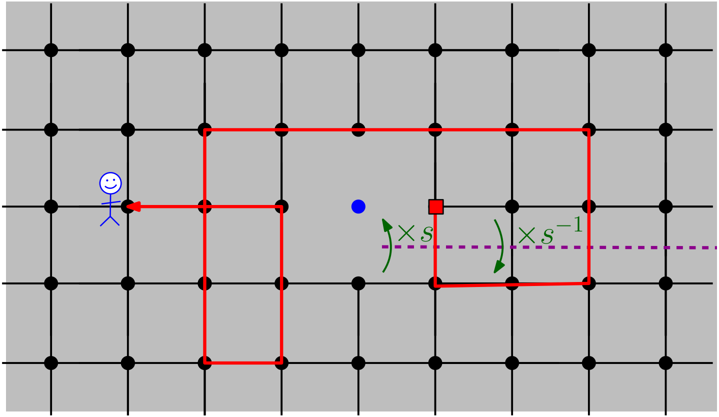
$$A(\omega) = e^{-2i\kappa}A(\omega - \omega_2).$$

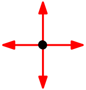
Other information $A(\omega) - x(\omega)$ has no poles in $\Omega_{-4}, \Omega_{-3}, \ldots, \Omega_0$.
Solution: Explicit in terms of the Jacobi theta function

$$A\left(\frac{\omega_1}{\pi}z - \frac{\gamma}{2}\right) = -x_c\frac{\vartheta(\delta-\alpha,\tau)\vartheta(\delta+\gamma+\alpha,\tau)}{\vartheta(\gamma+2\delta,\tau)\vartheta(\kappa,\tau)}\left(\frac{\vartheta(z+\gamma+\delta+\kappa,\tau)}{\vartheta(z+\gamma+\delta,\tau)} - e^{-2i\kappa}\frac{\vartheta(z-\delta+\kappa,\tau)}{\vartheta(z-\delta,\tau)}\right).$$

Part 4: Special cases

Simple walks





Kernel:

$$K(x,y;t) = 1 - t\left(x + y + \frac{1}{x} + \frac{1}{y}\right)$$

Parametrisation of Kernel curve:

$$X(z) = e^{-i\gamma}\frac{\vartheta(z,\tau)\vartheta(z+\gamma,\tau)}{\vartheta(z-\gamma,\tau)\vartheta(z+2\gamma,\tau)} \quad \text{and} \quad Y(z) = e^{-i\gamma}\frac{\vartheta(z,\tau)\vartheta(z-\gamma,\tau)}{\vartheta(z+\gamma,\tau)\vartheta(z-2\gamma,\tau)},$$

where $\gamma = \frac{\pi\tau}{4}$ and τ is determined by

$$e^{-i\frac{\gamma}{2}}\frac{\vartheta\left(\frac{\gamma}{2},\tau\right)}{\vartheta\left(\frac{3\gamma}{2},\tau\right)} = \frac{\sqrt{1+4t}-1}{2\sqrt{t}}.$$

So $q := e^{i\gamma} = t + 4t^3 + 34t^5 + 360t^7 + \cdots$

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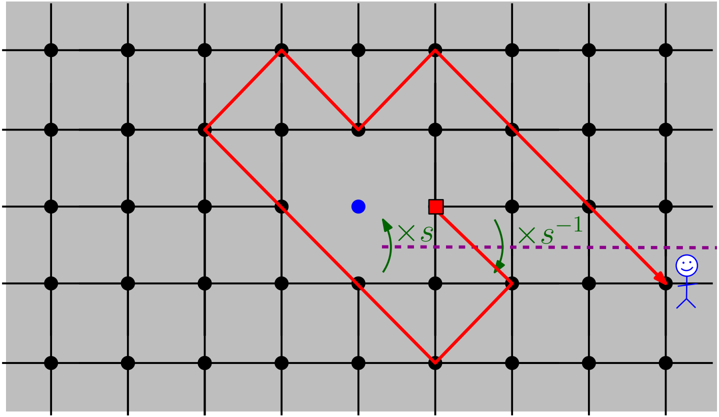
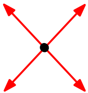
So $q := e^{i\gamma} = t + 4t^3 + 34t^5 + 360t^7 + \cdots$

Expression for excursions: (previously solved by [Budd, 2017])

$$\begin{aligned}\mathsf{E}(t,e^{2i\kappa}) &= \frac{t^{-1}e^{i\gamma}}{e^{-2i\kappa}-1}\frac{\vartheta(2\gamma)}{\vartheta(\kappa)}\left(2\frac{\vartheta(2\gamma+\kappa)}{\vartheta(2\gamma)} - \frac{e^{-2i\kappa}\vartheta(\gamma-\kappa)+\vartheta(\gamma+\kappa)}{\vartheta(\gamma)}\right).\\ &= 1 + 3t^2 + 22t^4 + 211t^6 + (2308 + s + s^{-1})t^8 + \cdots\end{aligned}$$


$\times s$
$\times s^{-1}$



Parametrisation of Kernel curve:

$$X(z) = ie^{-2i\gamma}\frac{\vartheta(z+\frac{\gamma}{2}+\frac{\pi}{4},\tau)\vartheta(z+\frac{\gamma}{2}-\frac{\pi}{4},\tau)}{\vartheta(z-\frac{3}{2}\gamma-\frac{\pi}{4},\tau)\vartheta(z+\frac{5}{2}\gamma+\frac{\pi}{4},\tau)},$$

$$Y(z) = -ie^{-2i\gamma}\frac{\vartheta(z-\frac{\gamma}{2}+\frac{\pi}{4},\tau)\vartheta(z-\frac{\gamma}{2}-\frac{\pi}{4},\tau)}{\vartheta(z+\frac{3}{2}\gamma-\frac{\pi}{4},\tau)\vartheta(z-\frac{5}{2}\gamma+\frac{\pi}{4},\tau)},$$

where $\gamma = \frac{\pi\tau}{4}$ and τ is determined by

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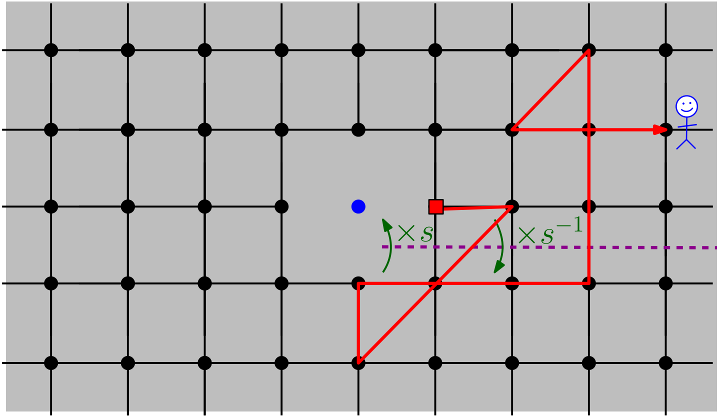
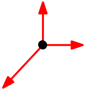
So $q := e^{i\gamma} = t + 4t^3 + 34t^5 + 360t^7 + \cdots$

Expression for excursions: [Budd, 2017]

$$\mathsf{E}(t,e^{2i\kappa}) = \frac{-e^{2i\gamma}}{t(1-e^{-2i\kappa})}\frac{\vartheta(2\gamma+\kappa,\tau)\vartheta(2\gamma-\frac{\pi}{2},\tau)-\vartheta(2\gamma,\tau)\vartheta(2\gamma-\frac{\pi}{2}+\kappa,\tau)}{\vartheta(\frac{\pi}{2},\tau)\vartheta(\kappa,\tau)}$$

$$= 1 + 4t^2 + \left(\frac{1}{s}+34+s\right)t^4 + \left(\frac{20}{s}+360+20s\right)t^6 + \cdots\cdots$$


$\times s$
$\times s^{-1}$



Parametrisation of Kernel curve:

$$X(z) = e^{-\frac{2}{3}i\gamma}\frac{\vartheta(z+\frac{\gamma}{2},\tau)^2}{\vartheta(z-\frac{\gamma}{2},\tau)\vartheta(z+\frac{3}{2}\gamma,\tau)},$$

$$Y(z) = e^{-\frac{2}{3}i\gamma}\frac{\vartheta(z-\frac{\gamma}{2},\tau)^2}{\vartheta(z+\frac{\gamma}{2},\tau)\vartheta(z-\frac{3}{2}\gamma,\tau)},$$

where $\gamma = \frac{\pi\tau}{6}$ and τ is determined by

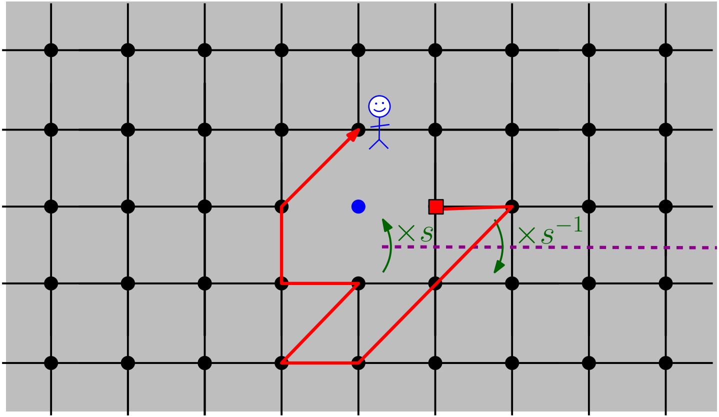
$$-2e^{-\frac{2}{3}i\gamma}\frac{\vartheta(\frac{\gamma}{2},\tau)}{\vartheta(\frac{3}{2}\gamma,\tau)} + e^{\frac{4}{3}i\gamma}\frac{\vartheta(\frac{3\gamma}{2},\tau)^2}{\vartheta(\frac{1}{2}\gamma,\tau)^2} = \frac{1}{t}.$$

So $q := e^{\frac{2i\gamma}{3}} = t + 5t^4 + 68t^7 + 1188t^{10} + \cdots$.

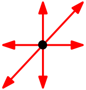
Expression for excursions:

$$\mathsf{E}(t,e^{2i\kappa}) = e^{\frac{2}{3}i\gamma}\frac{\vartheta(\gamma+\kappa)\vartheta'(\gamma) - \vartheta(\gamma)\vartheta'(\gamma+\kappa) + e^{-2i\kappa}\vartheta(\gamma-\kappa)\vartheta'(\gamma) - e^{-2i\kappa}\vartheta(\gamma)\vartheta'(\gamma-\kappa)}{t(1-e^{-2i\kappa})\vartheta(\kappa,\tau)\vartheta'(0,\tau)}$$

$$= 1 + 4t^3 + 48t^6 + (s^{-1} + 770 + s)t^9 + \cdots$$



$\times s$
$\times s^{-1}$



Parametrisation of Kernel curve:

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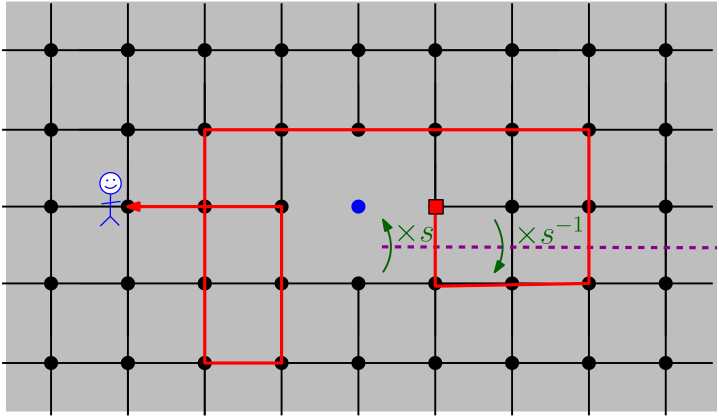
where $\gamma=\frac{\pi\tau}{6}$ and τ is determined by

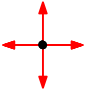
$$e^{-4i\gamma}\frac{\vartheta(\gamma,\tau)^3}{\vartheta(3\gamma,\tau)^3}=\frac{t}{1+2t},$$

So $q:=e^{2i\gamma}=t+t^2+7t^3+23t^4+122t^5+554t^6+\cdots$.

Expression for excursions:

$$\mathsf{E}(t,e^{2i\kappa})=-\frac{\left(2\vartheta\left(2\gamma\right)\vartheta\left(3\gamma+\kappa\right)-\vartheta\left(3\gamma\right)\vartheta\left(2\gamma+\kappa\right)-e^{-2i\kappa}\vartheta\left(3\gamma\right)\vartheta\left(2\gamma-\kappa\right)\right)e^{2i\gamma}}{t\left(1-e^{-2i\kappa}\right)\vartheta\left(\gamma\right)\vartheta\left(\kappa\right)}$$
$$=1+5t^2+8t^3+62t^4+216t^5+(s^{-1}+1199+s)t^6+\cdots$$


$\times s$
$\times s^{-1}$



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So $q := e^{i\gamma} = t + 4t^3 + 34t^5 + 360t^7 + \cdots$

Unrestricted walks ($x = y = 1$): Let $\lambda = \frac{1-2t-\sqrt{1-4t}}{2t}$.

$$\mathsf{W}(1,1;t,e^{2i\kappa}) = \frac{e^{i\gamma}}{(1-4t)(\lambda+1)}\frac{\vartheta(2\gamma)}{\vartheta(\kappa)}\left(\frac{\vartheta(\frac{\pi}{2}+\frac{5\gamma}{2}+\kappa)}{\lambda\vartheta(\frac{\pi}{2}+\frac{5\gamma}{2}+\kappa)} - e^{-2i\kappa}\frac{\vartheta(\frac{\pi}{2}+\frac{5\gamma}{2}+\kappa)}{\vartheta(\frac{\pi}{2}+\frac{5\gamma}{2}-\kappa)} - e^{-}\right.$$

$$= 1 + (2+s^{-1})t + (8+4s^{-1})t^2 + \cdots$$

Kernel:

$$K(x,y;t) = 1 - t\left(x + y + \frac{1}{x} + \frac{1}{y}\right)$$

Parametrisation of Kernel curve:

$$X(z) = e^{-i\gamma}\frac{\vartheta(z,\tau)\vartheta(z+\gamma,\tau)}{\vartheta(z-\gamma,\tau)\vartheta(z+2\gamma,\tau)} \quad \text{and} \quad Y(z) = e^{-i\gamma}\frac{\vartheta(z,\tau)\vartheta(z-\gamma,\tau)}{\vartheta(z+\gamma,\tau)\vartheta(z-2\gamma,\tau)},$$

where $\gamma = \frac{\pi\tau}{4}$ and τ is determined by

$$e^{-i\frac{\gamma}{2}}\frac{\vartheta\left(\frac{\gamma}{2},\tau\right)}{\vartheta\left(\frac{3\gamma}{2},\tau\right)} = \frac{\sqrt{1+4t}-1}{2\sqrt{t}}.$$

So $q := e^{i\gamma} = t + 4t^3 + 34t^5 + 360t^7 + \cdots$

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Corollary (excursions): Let E_n be a random variable given by the winding angle θ of a random excursion of length $2n$

Theorem: $\frac{\pi E_n}{\log(n)}$ converges to a random variable E with density

$$f_E(x) = \frac{(x-1)e^x + (x+1)e^{-x}}{(e^x - e^{-x})^3}.$$

Corollary (unrestricted walks): Let W_n be a random variable given by the winding angle θ of a random *walk* of length n

Theorem: $\frac{\pi W_n}{\log(n)}$ converges to a random variable W with density

$$f_W(x) = \frac{2}{(e^x + e^{-x})^2}.$$

Recall: Affected by $(0,0)$ being forbidden - this makes random walks stay away from $(0,0)$ and not wind as much.

Further Problems

- Same principle for other models e.g., brownian motion on non-convex cones (See [Fayolle,Franceschi,Raschel,2023])

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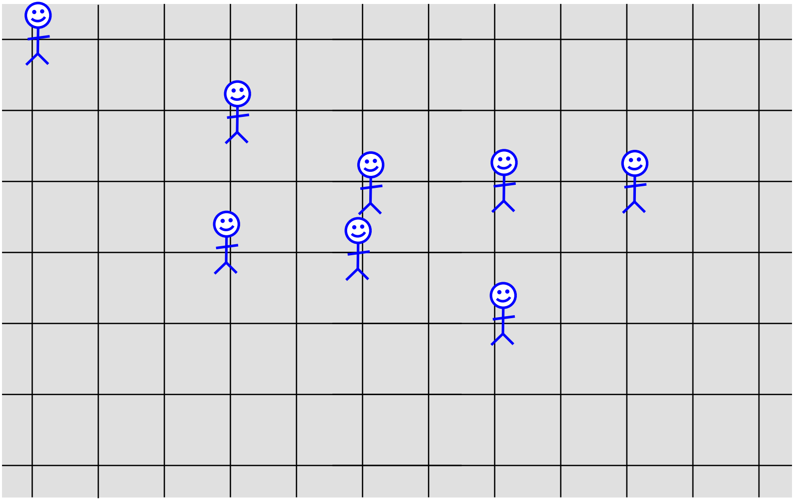
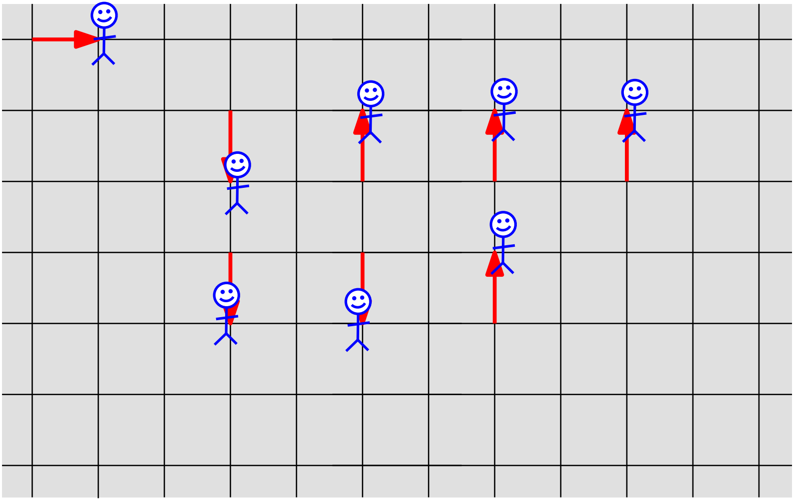
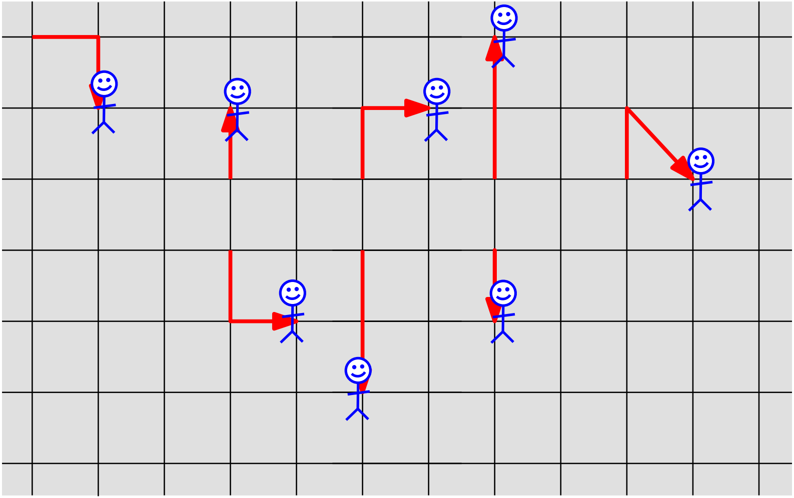
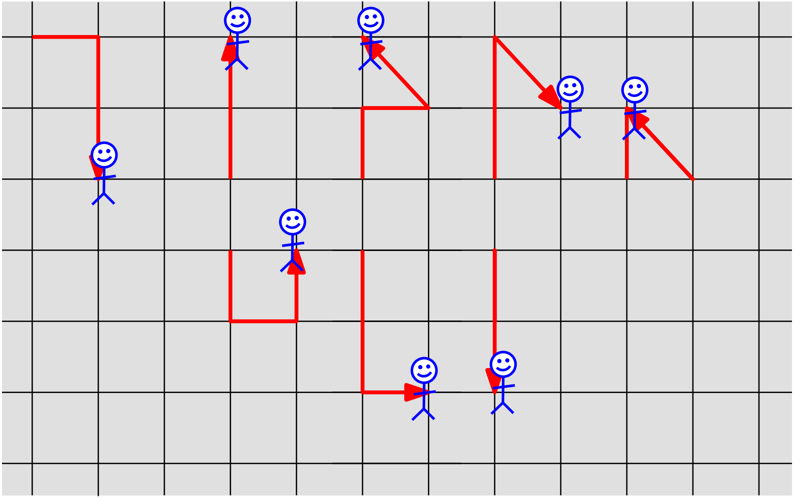
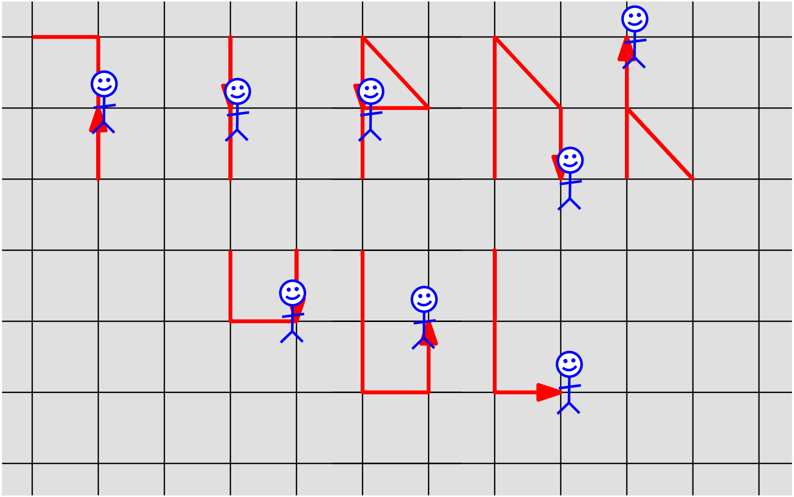
- Walks by winding angle with larger steps?
- Walks winding around multiple points: What's a good model?

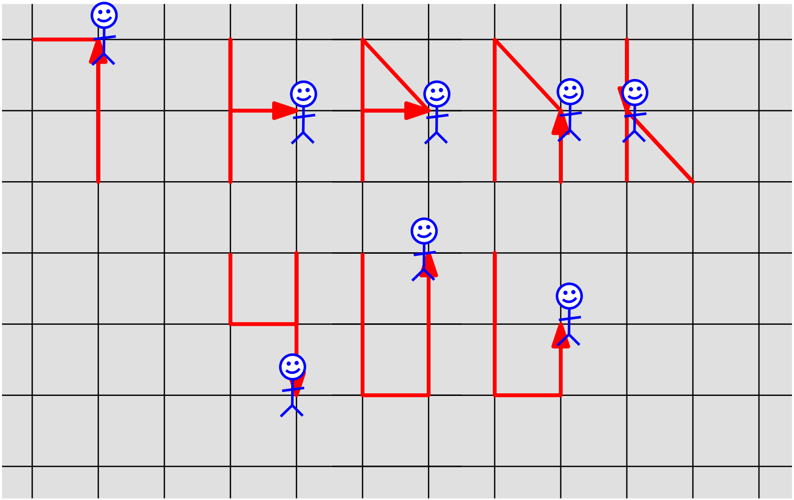


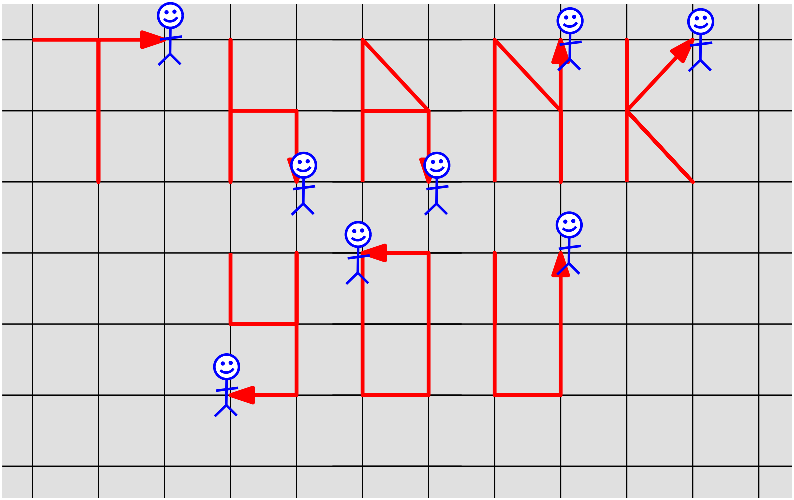












BONUS SLIDE: PARAMETERIZATION OF $K(x,y)=0$

Write $K(x,y)=A(x)y^2+B(x)y+C(x)$, then

$$Y(x)=\frac{-B(x)\pm\sqrt{B(x)^2-4A(x)C(x)}}{2A(x)}$$

parameterizes $K(x,Y(x))=0$. Typically, $Y_+(x)$ is meromorphic on:

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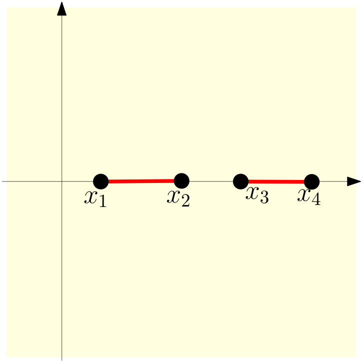

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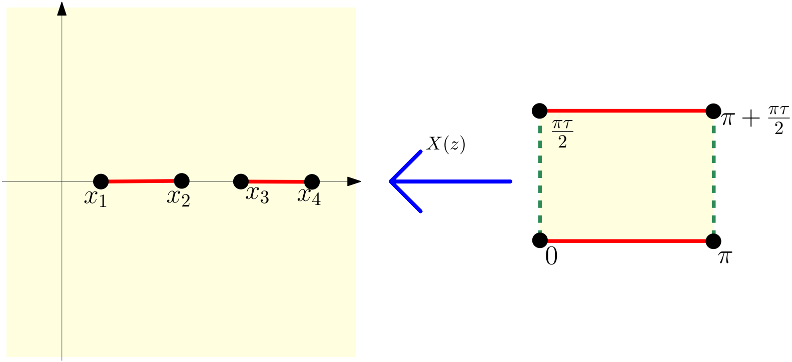
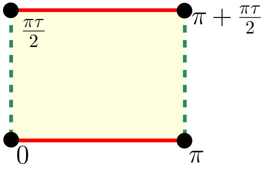


Bonus slide: Parameterization of $K(x,y)=0$



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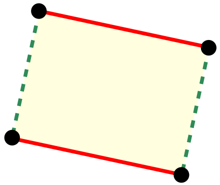

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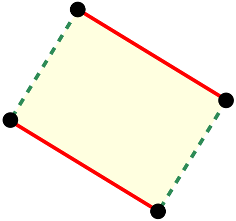

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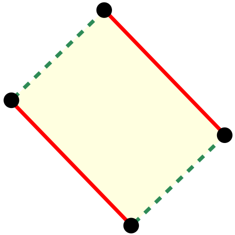

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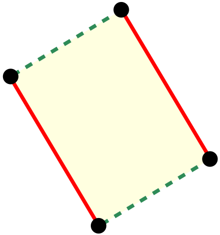

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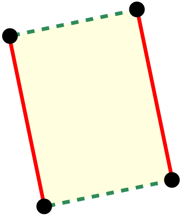

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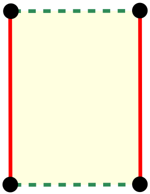

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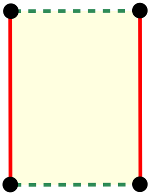

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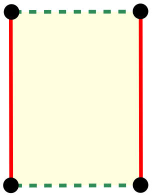

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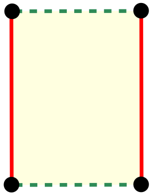

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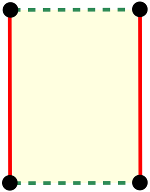

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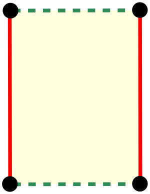

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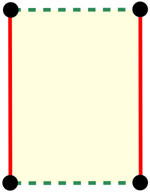

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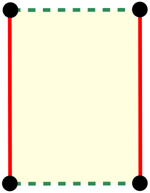

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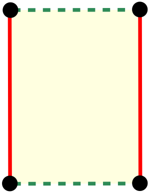

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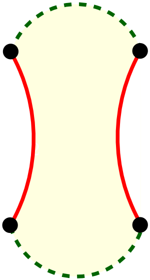

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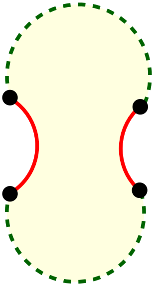

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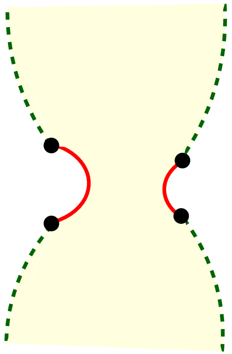

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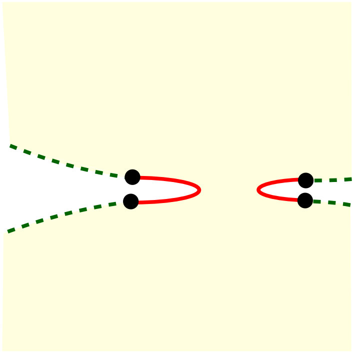

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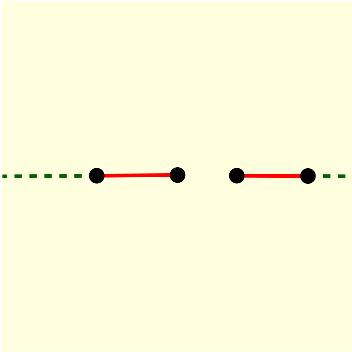

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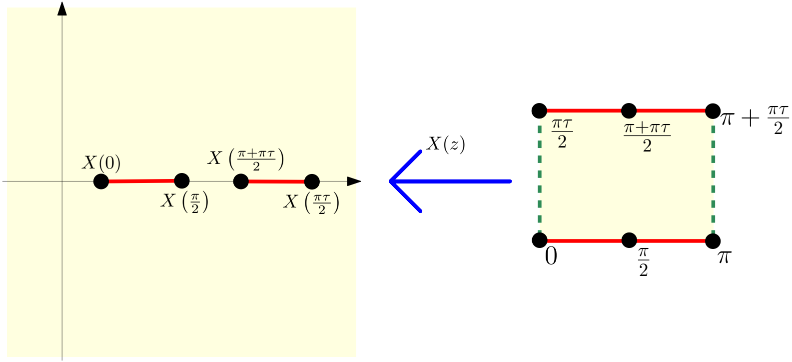

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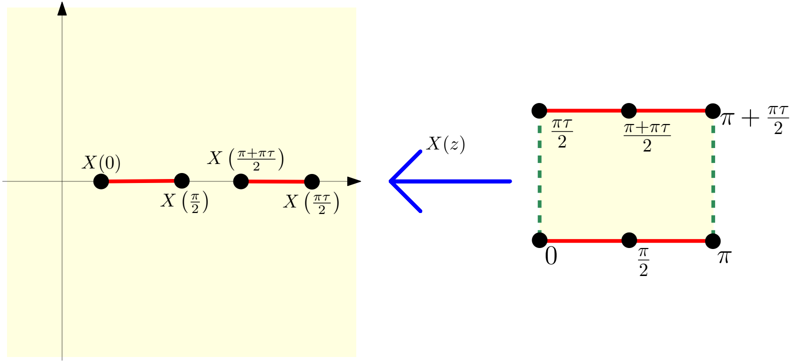
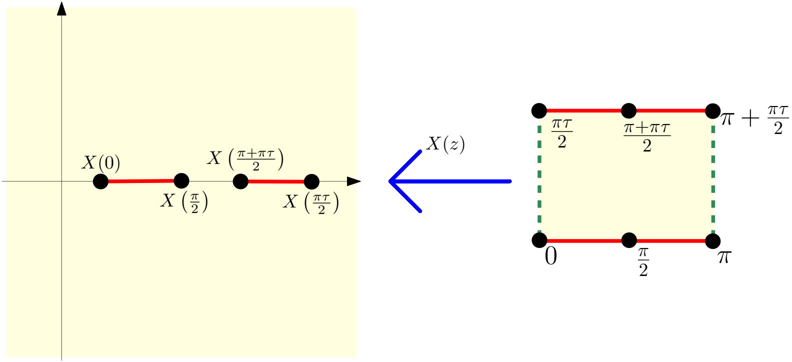


By symmetry, for $r \in \mathbb{R}$:

- $X(r) = X(\pi - r) = X(-r)$
- $X(\frac{\pi\tau}{2} + r) = X(\frac{\pi\tau}{2} - r)$

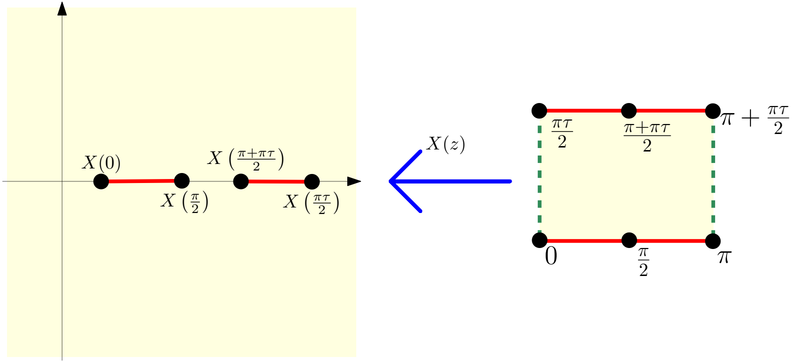
Bonus slide: Parameterization of $K(x,y)=0$



For $z \in \mathbb{C}$:

- $X(z) = X(\pi - z) = X(-z)$
- $X(z) = X(\pi\tau - z)$

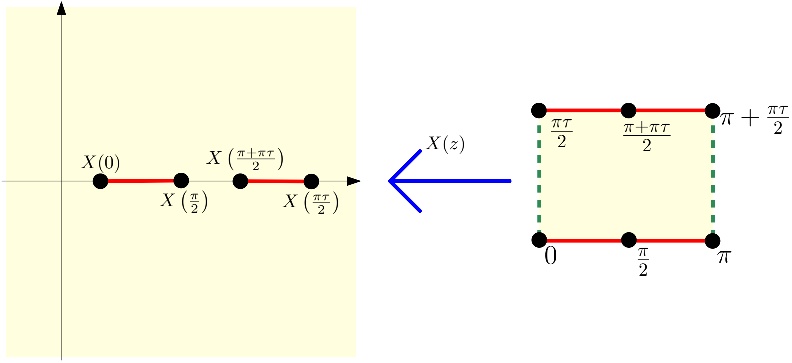
BONUS SLIDE: PARAMETERIZATION OF $K(x,y)=0$



For $z \in \mathbb{C}$:

- $X(z) = X(\pi - z) = X(-z) = X(\pi\tau + z)$
- $X(z) = X(\pi\tau - z)$

Bonus slide: Parameterization of $K(x, y) = 0$

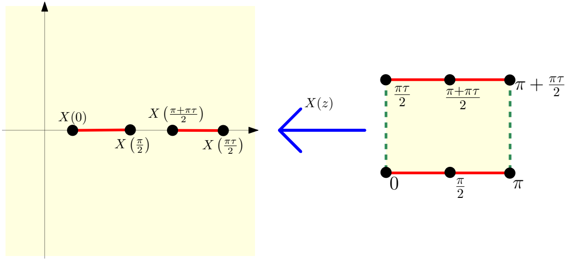


For $z \in \mathbb{C}$:

- $X(z) = X(\pi - z) = X(-z) = X(\pi\tau + z)$

$$X(z) = c\frac{\vartheta(z-\alpha)\vartheta(z+\alpha)}{\vartheta(z-\beta)\vartheta(z+\beta)}$$

Bonus slide: Parameterization of $K(x,y)=0$



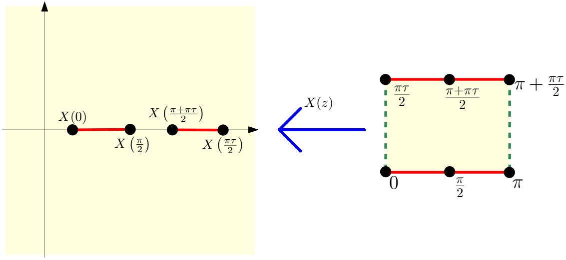
Recall:

$$y(x)=\frac{-B(x)\pm\sqrt{B(x)^2-4A(x)C(x)}}{2A(x)}.$$

Consider $Y(z)=y(X(z))$. By symmetry, for $r\in\mathbb{R}$:

- $X(r)=X(-r)$, so $Y(r)+Y(-r)=-\frac{B(X(r))}{A(X(r))}$.
- Similarly, $Y\left(\frac{\pi\tau}{2}+r\right)+Y\left(\frac{\pi\tau}{2}-r\right)=-\frac{B\left(X\left(\frac{\pi\tau}{2}+r\right)\right)}{A\left(X\left(\frac{\pi\tau}{2}+r\right)\right)}$.

Bonus slide: Parameterization of $K(x, y) = 0$



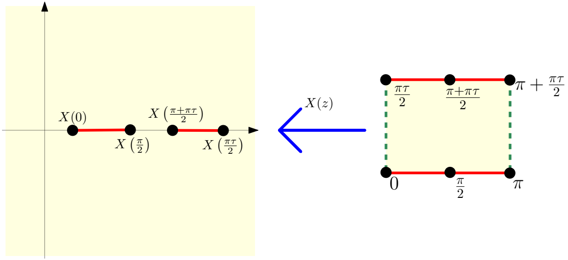
Recall:

$$y(x) = \frac{-B(x) \pm \sqrt{B(x)^2 - 4A(x)C(x)}}{2A(x)}.$$

Consider $Y(z) = y(X(z))$. For $z \in \mathbb{C}$:

- $Y(z) + Y(-z) = -\frac{B(X(z))}{A(X(z))}$.
- $Y(z) + Y(\pi\tau - z) = -\frac{B(X(z))}{A(X(z))}$.

BONUS SLIDE: PARAMETERIZATION OF $K(x,y)=0$



For $z \in \mathbb{C}$:

- $Y(z)+Y(-z) = -\frac{B(X(z))}{A(X(z))}$.
- $Y\left(z\right)+Y\left(\pi\tau - z\right) = -\dfrac{B\left(X\left(z\right)\right)}{A\left(X\left(z\right)\right)}$.

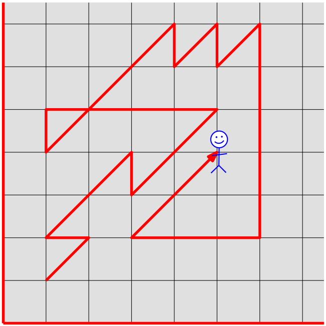
So $Y(z)=Y(z+\pi\tau)=Y(z+\pi)$

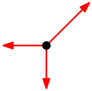
$$\Rightarrow Y(z) = c\frac{\vartheta(z-\gamma)\vartheta(z-\delta)}{\vartheta(z-\epsilon)\vartheta(z-\gamma-\delta+\epsilon)}.$$

Part 5: Relation to walks in cones

The model: Choose a step set $S \in \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$.
Problem: Determine the number $q_{i,j,n}$ of paths in the positive quadrant with steps in S from $(1, 1)$ to (i, j) of length n.
Equivalently: Determine the generating function

$$\mathsf{Q}(x, y; t) := \sum_{n\geq 0} \sum_{i,j\geq 1} q_{i,j,n} t^n x^i y^j .$$





Walks with small steps in the quarter plane

The model: Choose a step set $S \in \{-1,0,1\}^2 \setminus \{(0,0)\}$.
Problem: Determine the number $q_{i,j,n}$ of paths in the positive quadrant with steps in S from $(1,1)$ to (i,j) of length n.
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$$\mathsf{Q}(x,y;t) := \sum_{n\geq 0}\sum_{i,j\geq 1} q_{i,j,n} t^n x^i y^j.$$

Systematic approach: 79 distinct non-trivial step sets identified [Bousquet-Mélou, Mishna, 2010].

All models now classified using many methods

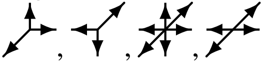
- Algebraic methods [Malyshev, Bousquet-Mélou, Mishna]
- Asymptotic analyses [Denisov, Wachtel, Mishna, Rechnitzer]
- Computer algebra [Bostan, Chyzak, Van Hoeij, Kauers, Pech]
- Galois Theory [Dreyfus, Hardouin, Roques, Singer]
- Analytic approach [Fayolle, Raschel, Kurkova, Bernardi]

Quadrant walks

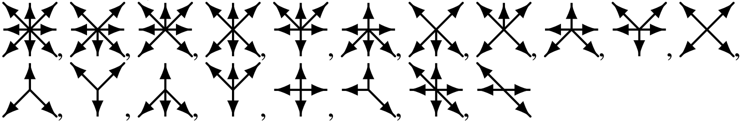
In total: 79 different non-trivial step sets S.

Generating function $\mathsf{Q}(x, y; t)$ is...

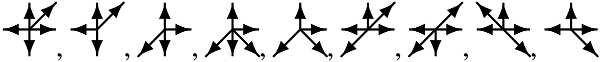
- Algebraic in 4 cases:



- D-finite in 19 further cases:



- D-algebraic in 9 further cases:



- In remaining 47 cases, $\mathsf{Q}(x, y; t)$ is not D-algebraic.

[Bousquet-Mélou, Mishna, Denisov, Wachtel, Rechnitzer, Bostan, Chyzak, Van Hoeij, Kauers, Pech, Dreyfus, Hardouin, Roques, Singer, Fayolle, Raschel, Kurkova, Bernardi]

In total: 79 different non-trivial step sets S.

Generating function $\mathsf{Q}(x, y; t)$ is...

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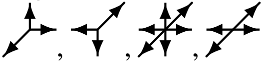

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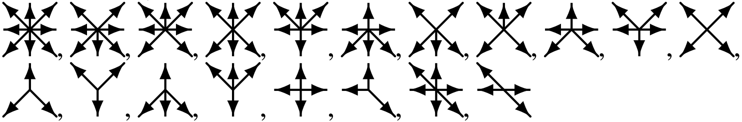
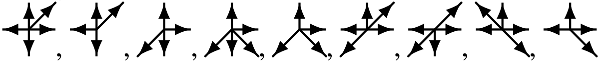


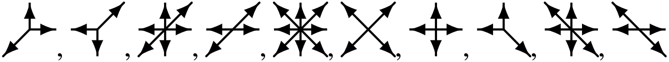
- D-algebraic in 9 further cases:



- In remaining 47 cases, $\mathsf{Q}(x, y; t)$ is not D-algebraic.

[E.P., 2022]: Same nature (in x) for 3/4-plane walks.

Concrete relation: The generating function for quadrant walks using any of the following step sets



can be derived from expressions for walks by winding angle using the reflection principal.

WALKS BY WINDING → WALKS IN CONES

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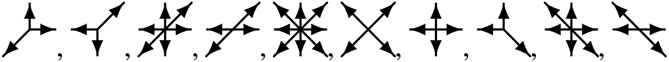

can be derived from expressions for walks by winding angle using the reflection principal.

Question: Is there an *interesting* classification of step sets for walks by winding angle, analogous to that of walks in cones?

Difficulty: $\mathsf{W}(x, y; t, s)$ itself is always D-algebraic but never D-finite.

BONUS SLIDE: PARAMETERIZATION OF $K(x,y)=0$

Equation characterising $Q(x,y) \equiv Q(t,x,y)$ for quadrant walks:

$$K(x,y)Q(x,y) + R(x,y) = 0.$$

$K(x,y)=0$ is parameterised by

$$X(z) = c_1 \frac{\vartheta(z-\alpha_1)\vartheta(z-\beta_1)}{\vartheta(z-\gamma_1)\vartheta(z-\delta_1)} \quad \text{and} \quad Y(z) = c_2 \frac{\vartheta(z-\alpha_2)\vartheta(z-\beta_2)}{\vartheta(z-\gamma_2)\vartheta(z-\delta_2)},$$

where the constants satisfy $\alpha_j + \beta_j = \gamma_j + \delta_j$ for $j = 1, 2$.
So, $R(X(z), Y(z)) = 0$.

BONUS SLIDE: PARAMETERIZATION OF $K(x, y) = 0$

In general: $K(x, y) = 0$ is parameterised by

$$X(z) = c_1 \frac{\vartheta(z - \alpha_1)\vartheta(z - \beta_1)}{\vartheta(z - \gamma_1)\vartheta(z - \delta_1)} \quad \text{and} \quad Y(z) = c_2 \frac{\vartheta(z - \alpha_2)\vartheta(z - \beta_2)}{\vartheta(z - \gamma_2)\vartheta(z - \delta_2)},$$

with $\alpha_j + \beta_j = \gamma_j + \delta_j$ for $j = 1, 2$.

BONUS SLIDE: PARAMETERIZATION OF $K(x, y) = 0$

For Kreweras paths:

$$Q(x, y) = 1 + xytQ(x, y) + \frac{t}{x}\left(Q(x, y) - Q(0, y)\right) + \frac{t}{y}\left(Q(x, y) - Q(x, 0)\right).$$

Then $K(x, y) = xy - tx^2y^2 - tx - ty = 0$ is parameterised by

$$X(z) = c_1 \frac{\vartheta(z - \alpha_1)\vartheta(z - \beta_1)}{\vartheta(z - \gamma_1)\vartheta(z - \delta_1)} \quad \text{and} \quad Y(z) = c_2 \frac{\vartheta(z - \alpha_2)\vartheta(z - \beta_2)}{\vartheta(z - \gamma_2)\vartheta(z - \delta_2)},$$

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with $\alpha_j+\beta_j=\gamma_j+\delta_j$ for $j=1,2$.

- $K(0,0)=0$, so WLOG $\alpha_1=\alpha_2=0$.

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- as $x\to 0$, we have $y(x)\sim -x$ or $y(x)\sim -\frac{1}{x^2}$, so $Y(z)$ has a double pole at $z=\beta_1$.

Bonus slide: Parameterization of $K(x,y)=0$

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with $\alpha_j + \beta_j = \gamma_j + \delta_j$ for $j = 1, 2$.

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- So $3\beta_1 = \pi\tau$.

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$$X(z) = \frac{e^{-\frac{4\pi\tau i}{9}}\vartheta(z)\vartheta\left(z-\frac{\pi\tau}{3}\right)}{\vartheta\left(z+\frac{\pi\tau}{3}\right)\vartheta\left(z-\frac{2\pi\tau}{3}\right)} \quad \text{and} \quad Y(z) = \frac{e^{-\frac{4\pi\tau i}{9}}\vartheta(z)\vartheta\left(z+\frac{\pi\tau}{3}\right)}{\vartheta\left(z-\frac{\pi\tau}{3}\right)\left(z+\frac{2\pi\tau}{3}\right)},$$

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Bonus slide: Parameterization of $K(x,y)=0$

Then $K(x,y) = xy - tx^2y^2 - tx - ty = 0$ is parameterised by

$$X(z) = \frac{e^{-\frac{4\pi\tau i}{3}}\vartheta(z,3\tau)\vartheta\left(z-\pi\tau,3\tau\right)}{\vartheta\left(z+\pi\tau,3\tau\right)\vartheta\left(z-2\pi\tau,3\tau\right)} \quad \text{and} \quad Y(z) = X(z+\pi\tau),$$

where

$$t = \frac{1}{X(z)Y(z) + X(z)^{-1} + Y(z)^{-1}}.$$

BONUS SLIDE: PARAMETERIZATION OF $K(x,y)=0$

Then $K(x,y)=xy-tx^2y^2-tx-ty=0$ is parameterised by

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where

$$t=e^{-\frac{\pi\tau i}{3}}\frac{\vartheta'(0,3\tau)}{4i\vartheta(\pi\tau,3\tau)+6\vartheta'(\pi\tau,3\tau)}.$$