Couplings and attractiveness for general exclusion processes

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Un quart de siècle pour un quart de plan
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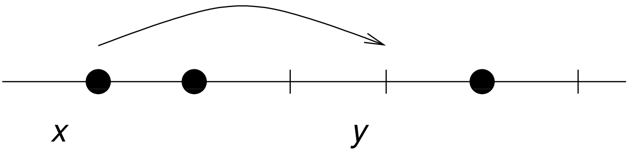
Outline

Basic example : The simple exclusion process (SEP)

[Liggett]



conservative dynamics :

- At most one particle per site : for $z \in \mathbb{Z}$, $\eta(z) = 0$ or 1.
- From each site x, choice of y with $p(y - x)$.
- According to (independent) exponential clocks, jump from x to y if possible (*exclusion rule*), that is $\eta(x) = 1, \eta(y) = 0$. No other condition on η.

Attractiveness [Liggett]

$(\eta_t)_{t\geq 0}$: an interacting particle system of state space $\Omega = X^S$, with $X \subset \mathbb{Z}$, $S = \mathbb{Z}^d$. It is a Markov process with generator L and semi-group $T(t)$.

- Partial order on Ω :
  $\forall \xi, \zeta \in \Omega, \quad \xi \leq \zeta \Longleftrightarrow (\forall x \in S, \quad \xi(x) \leq \zeta(x))$
  $f : \Omega \to \mathbb{R}$ is monotone if : $\forall \xi, \zeta \in \Omega,\ \xi \leq \zeta \Longrightarrow f(\xi) \leq f(\zeta)$
  $\mathcal{M} = \{$ bounded, monotone, continuous functions on $\Omega\}$
- Stochastic order on probability measures $\mathcal{P}(\Omega)$:
  $\forall \nu, \nu' \in \mathcal{P}(\Omega), \quad \nu \leq \nu' \Longleftrightarrow (\forall f \in \mathcal{M}, \nu(f) \leq \nu'(f))$.
- $(\eta_t)_{t\geq 0}$ is attractive if the following equivalent conditions are satisfied.
  (a) $f \in \mathcal{M}$ implies $T(t)f \in \mathcal{M}$ for all $t \geq 0$.
  (b) For $\nu, \nu' \in \mathcal{P}(\Omega)$, $\nu \leq \nu'$ implies $\nu T(t) \leq \nu' T(t)$ for all $t \geq 0$.

Properties of SEP :

- Basic coupling preserves order $\rightsquigarrow$ SEP is attractive
- $(\mathcal{I} \cap \mathcal{S})_e = \{\nu_\rho,\ \rho \in [0, 1]\}$, Bernoulli product measures

Extension to other conservative dynamics ?

- Multiple particle jumps : $K \geq 1$ particles per site, $k \geq 1$ particles attempt to jump from x to y.
  Generalized misanthropes processes (MMP)
  [Gobron, Saada], [Fajfrová, Gobron, Saada]
  Conditions on the rates to have product invariant measures.
- Speed change exclusion : This talk

Extension to non conservative dynamics ? [Borrello]

A general exclusion process

$S = \mathbb{Z}^d$; $(\eta_t)_{t\geq 0}$ Markov process on $\Omega = \{0,1\}^S$ with semi-group $(T(t), t \geq 0)$ and infinitesimal generator :

$$\mathcal{L}f(\eta) = \sum_{x,y\in S} \eta(x)(1-\eta(y))\Gamma_\eta(x,y)\left[f(\eta^{x,y}) - f(\eta)\right], \qquad (1)$$

with $\Gamma_\eta(x,y)$ independent of $\eta(x)$ and $\eta(y)$, and $\eta^{x,y}$ obtained from η by exchanging the occupation numbers at sites x and y.

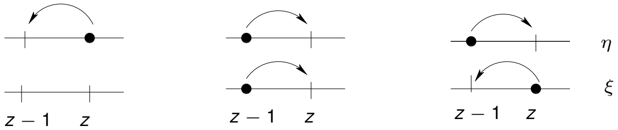
- Existence assumption :

$$\sup_{v\in S}\sum_{u\in S}\sup_{\eta\in\Omega}\Gamma_\eta(u,v) < +\infty \;\text{ and }\; \sup_{u\in S}\sum_{v\in S}\sup_{\eta\in\Omega}\Gamma_\eta(u,v) < +\infty.$$

- Conservative dynamics : $\eta(x)+\eta(y)$ is a *conserved quantity* in a transition.
- When $\Gamma_\eta(x,y)$ is independent of η, and $\Gamma_\eta(x,y) = p(x,y)$, one recovers SEP.

Coupling for SEP

Particles jump together as much as possible : for both marginals, jumps have the same departure site and the same arrival site ; this is basic coupling.



- (a) only possible : clock rings at z ; η-marginal moves, at rate q ;
  $(0,1) \geq (0,0)$ gives $(1,0) \geq (0,0)$: order preserved.
- (b) clock rings at $z-1$, η and ξ both move at rate p ; order preserved, correct marginals : no other arrow needed.
- (c) no order, thus no condition ; clock rings at $z-1$ or z ; marginals independent, and correct (rates p, q).

conclusion

There is a discrepancy at z if $\eta(z) \neq \xi(z)$.

- basic coupling decreases discrepancies, and cannot create new ones.

Invariant measures for SEP

Skeleton of proof of $(\mathcal{I}\cap\mathcal{S})_e = \{\nu_\rho,\ \rho\in[0,1]\}$ [Liggett]
Let $\rho\in[0,1]$ and $\mu\in(\mathcal{I}\cap\mathcal{S})_e$. there exists $\overline{\nu}\in\left(\overline{\mathcal{I}}\cap\overline{\mathcal{S}}\right)_e$ (for the coupled process, via basic coupling) with marginals ν_ρ and μ.
By attractiveness,

$$\forall(x,y)\in S^2,\ \overline{\nu}\{(\xi,\zeta):\xi(x)>\zeta(x)\ \text{ and } \xi(y)<\zeta(y)\}=0 \quad (2)$$

Then $\overline{\nu}\{(\xi,\zeta):\xi\le\zeta\ \text{ or } \xi\ge\zeta\}=1$,
so that $\mu=\nu_{\rho_0}$, where $\rho_0=\sup\{\rho,\mu\ge\nu_\rho\}$.

main point : to derive (2) :
Given any pair of discrepancies of opposite signs, bring them close to one another (in the sense of $p(.,.)$) using an irreducibility condition on $p(.,.)$. They will merge due to basic coupling properties. $\rightsquigarrow$ discrepancies of opposite signs decrease and finally disappear.

SEP is the only exclusion process for which the basic coupling is monotone

For an exclusion process with general rates $\Gamma_\eta(x, y)$, assume monotonicity of the associated basic coupling.

Then jump rates are independent on the configuration, $\Gamma_\eta(x, y) = p(x, y)$ and the only exclusion process for which basic coupling is monotone is the SEP.

Attractiveness and coupling for general exclusion

Strategy : Six main steps :

1. Obtain necessary conditions on the transition rates for attractiveness.
2. Construct a Markovian coupling.
3. Show that it is increasing, that is it preserves the stochastic order between marginal configurations, under conditions from Step 1 (thus proving their sufficiency).
4. Modify it into a quasi-attractive Markovian coupling, that is, such that discrepancies between unordered configurations do not increase on the sites involved in a coupled transition.
5. Construct an attractive Markovian coupling, that is, under which any pair of discrepancies of opposite sign have a positive probability to disappear in finite time ; this also requires some irreducibility conditions.
6. Application : Determination of $(\mathcal{I} \cap \mathcal{S})_e$.

Step 6. Determination of $(\mathcal{I}\cap\mathcal{S})_e$

Let $(\eta_t)_{t\geq 0}$ be an exclusion process with generator (1).

Definition

An open edge (x,y) is s.t. $\Gamma_\xi(x,y)>0$, $\forall\xi\in\Omega$.
S is fully connected if $\forall(x,y)\in S^2$, $x\neq y$, $\exists\,\{x_0,\cdots,x_n\}$ for $n>0$ s.t. (x_{i-1},x_i) open for $i\in\{1,\cdots,n\}$ with either $x_0=x$ and $x_n=y$, or $x_0=y$ and $x_n=x$.

Theorem

Assume $(\eta_t)_{t\geq 0}$ has translation invariant jump rates, s.t. S is fully connected. If $(\eta_t)_{t\geq 0}$ is attractive, and the system satisfies an irreducibility condition, then
1) $(\mathcal{I}\cap\mathcal{S})_e=\{\mu_\rho,\ \rho\in\mathcal{R}\}$, with $\mathcal{R}$ a closed subset of $[0,1]$ containing $\{0,1\}$, and $\forall\rho\in\mathcal{R}$, μ_ρ is a translation invariant probability measure on Ω with $\mu_\rho[\eta(0)]=\rho$; $\mu_\rho\leq\mu_{\rho'}$ if $\rho\leq\rho'$;
2) if $(\eta_t)_{t\geq 0}$ possesses a one parameter family $\{\mu_\rho\}_\rho$ of product invariant and translation invariant probability measures, then $(\mathcal{I}\cap\mathcal{S})_e=\{\mu_\rho\}_\rho$. [Bahadoran et al.]

Step 1. Attractiveness conditions

Theorem

The exclusion process defined by (1) *is* monotone *iff* $\forall(\xi,\zeta) \in \Omega^2$ *such that* $\xi \leq \zeta$, *we have :*
1) $\forall y \in S$ *such that* $\zeta(y) = 0$,

$$\sum_{x\in S} \xi(x)\left[\Gamma_\xi(x,y) - \Gamma_\zeta(x,y)\right]^+ \leq \sum_{x\in S} \zeta(x)(1-\xi(x))\Gamma_\zeta(x,y), \quad (3)$$

2) $\forall x \in S$ *such that* $\xi(x) = 1$,

$$\sum_{y\in S}(1-\zeta(y))\left[\Gamma_\zeta(x,y) - \Gamma_\xi(x,y)\right]^+ \leq \sum_{y\in S} \zeta(y)(1-\xi(y))\Gamma_\xi(x,y). \quad (4)$$

Guideline : systems with denumerable state space. [Massey]

Interpretation of equations (3)–(4)

The l.h.s. of (3) measures the excess rate at which an empty site y is filled in the smaller configuration ξ, so that coupling jumps in both configurations from the same initial sites x to y will not be sufficient to preserve partial order if this sum is different from zero. Equation (3) suggests that partial order could be preserved by coupling such "excess rate" jumps with jumps involved in the r.h.s., that is jumps to y from sites occupied in configuration ζ, but empty in ξ.

Equation (3) states that such rates are sufficient to do so.

For equation (4) : similar reasoning.

Examples, with attractiveness conditions

A lattice gas model [Spitzer], [Liggett]

$$\Gamma_\eta(x, y) = q(x, y)c_\eta(x),$$

where $q : S \times S \to [0, +\infty)$ is a configuration independent jump intensity between sites x and y.satisfying

$$\forall x \in S,\ q(x, x) = 0; \qquad \sup_{x \in S} \sum_{y \in S} [q(x, y) + q(y, x)] < +\infty$$

and $c_\eta(x)$ is a configuration dependent velocity, satisfying

$$\sup_{x \in S, \eta \in X} c_\eta(x) < +\infty; \qquad \sup_{x \in S} \sum_{y \in S} \sup_{\eta \in X} |c_{\eta^y}(x) - c_\eta(x)| < +\infty,$$

where

$$\eta^y(z) = \begin{cases} 1 - \eta(y) & \text{if } z = y, \\ \eta(z) & \text{otherwise.} \end{cases}$$

The attractiveness conditions (3)–(4) read :
$\forall(\xi, \zeta) \in \Omega^2$ s.t. $\xi \leq \zeta$, one has
1) $\forall y \in S$ s.t. $\zeta(y) = 0$,

$$\sum_{x\in S} \xi(x)q(x,y)\left[c_\xi(x)-c_\zeta(x)\right]^+ \leq \sum_{x\in S} \zeta(x)(1-\xi(x))\,q(x,y)\,c_\zeta(x),$$

2) $\forall x \in S$ s.t. $\xi(x) = 1$,

$$\left[c_\zeta(x)-c_\xi(x)\right]^+ \sum_{y\in S}(1-\zeta(y))q(x,y) \leq c_\xi(x)\sum_{y\in S} \zeta(y)(1-\xi(y))q(x,y)$$

Sufficient conditions for attractiveness, which imply the above ones : [Liggett]
$\forall(\xi, \zeta) \in \Omega^2$ s.t. $\xi \leq \zeta$,
1) $\forall x \in S$, $c_\xi(x) \leq c_\zeta(x)$,
2) $\forall x \in S$ s.t. $\xi(x) = 1$,

$$\Big(\sum_{y\in S}(1-\zeta(y))q(x,y)\Big)c_\zeta(x) \leq \Big(\sum_{y\in S}(1-\xi(y))\,q(x,y)\Big)c_\xi(x).$$

Some one-dimensional examples beyond SEP

First : For SEP, the attractiveness conditions are empty since their l.h.s. gives 0.

- n.n.TA 2-step exclusion process : [Guiol]
  $\Gamma_\eta(x, y) = 1_{\{y=x+1\}} + 1_{\{y=x+2\}}\eta(x+1)$
- n.n.TA 2*-step exclusion process :
  $\Gamma_\eta(x, y) = 1_{\{y=x+1\}} + 1_{\{y=x+2\}}(1-\eta(x+1))$
- A range 2 traffic model : for $\alpha, \beta \in [0, 1]$,

  $$\Gamma_\eta(x, y) = 1_{\{y=x+1\}} + 1_{\{y=x+2\}}[\alpha\eta(x+1) + \beta(1-\eta(x+1))].$$

  $\beta = \alpha = 0$ corresponds to SEP, $\beta = 0, \alpha \neq 0$ to 2-step exclusion, and $\alpha = 0, \beta \neq 0$ to 2^*-step exclusion.

Proposition

The range 2 traffic model is attractive iff $|\beta - \alpha| \leq 1$.

A traffic model example

[Gray & Griffeath],[KLS]

For positive parameters $\alpha, \beta, \gamma, \delta$

$$\Gamma_\eta(x, x+1) = \begin{cases} \alpha & \text{if } \eta(x-1) = 1,\ \eta(x+2) = 0,\ \text{[accelerating]} \\ \beta & \text{if } \eta(x+2) = 1,\ \eta(x-1) = 0,\ \text{[braking]} \\ \gamma & \text{if } \eta(x-1) = \eta(x+2) = 1,\ \text{[congested]} \\ \delta & \text{if } \eta(x-1) = \eta(x+2) = 0,\ \text{[driving]}\ . \end{cases} \tag{5}$$

Not attractive, unless $\alpha = \beta = \gamma = \delta$ (SEP). But take a symmetrized version, i.e. with (5), with symmetric leftwards (6) :

$$\Gamma_\eta(x+1, x) = \begin{cases} \alpha & \text{if } \eta(x+2) = 1,\ \eta(x-1) = 0, \\ \beta & \text{if } \eta(x-1) = 1,\ \eta(x+2) = 0, \\ \gamma & \text{if } \eta(x+2) = \eta(x-1) = 1, \\ \delta & \text{if } \eta(x+2) = \eta(x-1) = 0. \end{cases} \tag{6}$$

Proposition

The symmetrized dynamics is attractive iff $\alpha, \beta, \gamma, \delta$ satisfy

$$\beta \leq \gamma \wedge \delta \leq \gamma \vee \delta \leq \alpha, \qquad \alpha \leq \beta + \gamma \wedge \delta, \qquad \delta \leq 2\beta. \tag{7}$$

The facilitated exclusion process has rates (5) with $\alpha = \gamma = 1$, $\beta = \delta = 0$. Hence it is not attractive, and its symmetrized version is not attractive either.
[Ayyer et al.],[Baik et al.],[Blondel et al.]

Guideline : systems with denumerable state space

[Massey] [Kamae et al.],[Kamae & Krengel]

Definition

W : a set endowed with a partial order relation.

$$V \subset W \text{ is increasing if : } \forall l \in V, m \in W, \quad l \leq m \Longrightarrow m \in V$$

$$V \subset W \text{ is decreasing if : } \forall l \in V, m \in W, \quad l \geq m \Longrightarrow m \in V$$

$$f : W \to \mathbb{R} \text{ is monotone if : } \forall l, m \in W, \quad l \leq m \Longrightarrow f(l) \leq f(m)$$

Examples. For $l \in W$, $I_l = \{m \in W : l \leq m\}$ is increasing ; $D_l = \{m \in W : l \geq m\}$ is decreasing.

Remark. $\forall V \subset W$,
V is increasing $\iff W \setminus V$ is decreasing $\iff \mathbf{1}_V$ is monotone

Sketch of proof of Theorem, necessary condition

$(\xi, \zeta) \in \Omega \times \Omega$, $\xi \leq \zeta$; $V \subset \Omega$ increasing cylinder set.

$$\text{If } \xi \in V \text{ or } \zeta \notin V, \quad \mathbf{1}_V(\xi) = \mathbf{1}_V(\zeta) \tag{8}$$

By attractiveness, since V increasing,

$$\forall t \geq 0, \ (T(t)\mathbf{1}_V)(\xi) \leq (T(t)\mathbf{1}_V)(\zeta) \tag{9}$$

Combining (8),(9),

$$\begin{aligned} t^{-1}[(T(t)\mathbf{1}_V)(\xi) - \mathbf{1}_V(\xi)] &\leq t^{-1}[(T(t)\mathbf{1}_V)(\zeta) - \mathbf{1}_V(\zeta)] \\ \text{thus when } t \to 0, \qquad (\mathcal{L}\mathbf{1}_V)(\xi) &\leq (\mathcal{L}\mathbf{1}_V)(\zeta) \end{aligned}$$

Applied for particular V's, $\zeta \notin V \Longrightarrow$(3) ; $\xi \in V \Longrightarrow$(4), since

$$\begin{aligned} (\mathcal{L}\mathbf{1}_V)(\xi) &= \sum_{\xi' \in \Omega} \Gamma(\xi, \xi')[\mathbf{1}_V(\xi') - \mathbf{1}_V(\xi)] \\ &= -\mathbf{1}_V(\xi) \sum_{\xi' \notin V} \Gamma(\xi, \xi') + \mathbf{1}_{V^c}(\xi) \sum_{\xi' \in V} \Gamma(\xi, \xi') \end{aligned}$$

Steps 2, 3. The Markovian increasing coupling

Proposition

For f on $\Omega \times \Omega$ *a cylinder function and* $(\xi, \zeta) \in \Omega \times \Omega$ *:*

$$\begin{aligned}
\overline{\mathcal{L}} f(\xi, \zeta) &= \sum_{x_1, y_1 \in S} \xi(x_1)(1 - \xi(y_1)) \left(\Gamma_\xi(x_1, y_1) - \varphi_{\xi,\zeta}(x_1, y_1)\right) \\
&\qquad \times \left(f(\xi^{x_1, y_1}, \zeta) - f(\xi, \zeta)\right) \\
&+ \sum_{x_2, y_2 \in S} \zeta(x_2)(1 - \zeta(y_2)) \left(\Gamma_\zeta(x_2, y_2) - \overline{\varphi}_{\xi,\zeta}(x_2, y_2)\right) \\
&\qquad \times \left(f(\xi, \zeta^{x_2, y_2}) - f(\xi, \zeta)\right) \\
&+ \sum_{\substack{x_1, y_1 \in S \\ x_2, y_2 \in S}} \xi(x_1)(1 - \xi(y_1))\zeta(x_2)(1 - \zeta(y_2)) G_{\xi,\zeta}(x_1, y_1; x_2, y_2) \\
&\qquad \times \left(f(\xi^{x_1, y_1}, \zeta^{x_2, y_2}) - f(\xi, \zeta)\right)
\end{aligned}$$

where $\forall(\xi,\zeta) \in \Omega \times \Omega$ *and* $\forall(x,y) \in S^2$,

$$\varphi_{\xi,\zeta}(x,y) := \sum_{x',y' \in S} \zeta(x')(1-\zeta(y'))G_{\xi,\zeta}(x,y;x',y'),$$
$$\overline{\varphi}_{\xi,\zeta}(x,y) := \sum_{x',y' \in S} \xi(x')(1-\xi(y'))G_{\xi,\zeta}(x',y';x,y).$$

Then $\overline{\mathcal{L}}$ generates a Markovian coupling between two copies of the Markov process defined by (1), provided that $\forall(\xi,\zeta) \in \Omega^2$ we have $G_{\xi,\zeta}(\cdots) \geq 0$ and

$$\forall(x_1,y_1) \in S^2, \varphi_{\xi,\zeta}(x_1,y_1) \leq \Gamma_\xi(x_1,y_1),$$
$$\forall(x_2,y_2) \in S^2, \overline{\varphi}_{\xi,\zeta}(x_2,y_2) \leq \Gamma_\zeta(x_2,y_2).$$

The form of the coupling rates

- $G_{\xi,\zeta}(x, y; x, y) = \Gamma_\xi(x, y) \wedge \Gamma_\zeta(x, y)$
- When $\zeta = \xi$, the only nonzero coupling rates are the diagonal terms $G_{\xi,\xi}(x, y; x, y) = \Gamma_\xi(x, y)$ so marginals remain equal.
- Uncoupled jumps unless $\xi \leq \zeta$ or $\xi > \zeta$. Then, $G_{\xi,\zeta}(x, y; x', y') \neq 0$ only if for the two coupled jumps either the same initial point, the same final point, or both.
- When $\xi \leq \zeta$ or $\xi > \zeta$: If $x = x'$, then $G_{\xi,\zeta}(x, y; x', y')$ is a combination of rates for y in the r.h.s. of (4) and y' in the l.h.s. of (4), with $y \neq y'$. Similar if $y = y'$.

Proposition

Under Conditions (3)–(4) *for attractiveness, this Markovian coupling is increasing.*

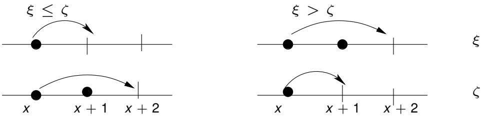
Steps 4, 5. The Markovian quasi-attractive and attractive couplings

- Generator $\overline{\mathcal{L}}^D$ with coupling rates $G^D_{\xi,\zeta}(x_1, y_1; x_2, y_2)$ built on $G_{\xi,\zeta}(x_1, y_1; x_2, y_2)$, for ξ, ζ not necessarily ordered, by involving $G_{\xi,\xi\vee\zeta}$ and $G_{\xi\vee\zeta,\zeta}$, where we do not necessarily have $x_1 = x_2$ or $y_1 = y_2$.
- New coupling rates : $\widetilde{G}_{\xi,\zeta}$, then $\widetilde{G}^D_{\xi,\zeta}$, also built using Conditions (3)–(4), which induce that, assuming some irreducibility condition (e.g. that all inequalities in (3) and (4) are strict), then for the coupled process $\left(\overline{\mathcal{I}} \cap \overline{\mathcal{S}}\right)_e \subset \{(\xi,\zeta) : \xi \leq \zeta\} \cup \{(\xi,\zeta) : \xi > \zeta\}$. In other words, discrepancies of opposite sign disappear, so that this coupling is attractive

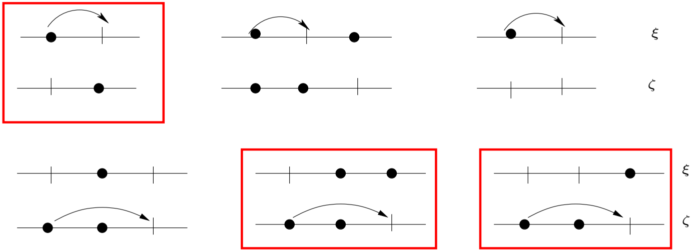
Couplings for 2 step exclusion

$$\Gamma_\eta(x, y) = 1_{\{y=x+1\}} + 1_{\{y=x+2\}}\eta(x+1)$$

Coupled jumps :



Uncoupled jumps for $\xi \leq \zeta$:



Couplings for 2 step exclusion

Due to the rates of 2 step exclusion, for the 4 constructed generators the transitions with non-negative rates are identical. The uncoupled jumps in a red border give a decrease of discrepancies. This induces

Theorem

$$(\mathcal{I} \cap \mathcal{S})_e = \{\nu_\rho,\ \rho \in [0, 1]\}$$

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Thank you for your Attention

SEP is the only exclusion process for which the basic coupling is monotone

For an exclusion process with general rates $\Gamma_\eta(x,y)$, assume monotonicity of the associated basic coupling. of generator

$$
\begin{aligned}
\widetilde{\mathcal{L}}f(\xi,\zeta) = & \sum_{x,y\in S} \xi(x)(1-\xi(y))\zeta(x)(1-\zeta(y))\Gamma_\xi(x,y)\wedge\Gamma_\zeta(x,y) \\
& \quad \times \big(f(\xi^{x,y},\zeta^{x,y}) - f(\xi,\zeta)\big) \\
+ & \sum_{x,y\in S} \xi(x)(1-\xi(y))\big[\Gamma_\xi(x,y) - \zeta(x)(1-\zeta(y))\Gamma_\xi(x,y)\wedge\Gamma_\zeta(x,y)\big] \\
& \quad \times \big(f(\xi^{x,y},\zeta) - f(\xi,\zeta)\big) \\
+ & \sum_{x,y\in S} \zeta(x)(1-\zeta(y))\big[\Gamma_\zeta(x,y) - \xi(x)(1-\xi(y))\Gamma_\xi(x,y)\wedge\Gamma_\zeta(x,y)\big] \\
& \quad \times \big(f(\xi,\zeta^{x,y}) - f(\xi,\zeta)\big). \qquad (10)
\end{aligned}
$$

Let ξ, ζ and $(x, y) \in S^2$ s.t. $\xi \leq \zeta$ and $\xi(x) = 1, \zeta(y) = 0$.

By monotonicity, rates of uncoupled jumps from x to y are $\equiv 0$ since otherwise order would be broken
either in x (if uncoupled jump of the ζ-particle)
or in y (if uncoupled jump of the ξ-particle).
$\Rightarrow$ by monotonicity of basic coupling, $\Gamma_\xi(x, y) = \Gamma_\zeta(x, y)$.

To extend this to any pair (ξ, ζ) whenever $\xi(x) = \zeta(x) = 1$ and $\xi(y) = \zeta(y) = 0$: Let $\sigma = \xi \wedge \zeta$;
then $\sigma(x) = 1$, $\sigma(y) = 0$ and both $\sigma \leq \xi$ and $\sigma \leq \zeta$.
Thus by above, $\Gamma_\sigma(x, y) = \Gamma_\xi(x, y) = \Gamma_\zeta(x, y)$.

Then jump rates are independent on the configuration, $\Gamma_\eta(x, y) = p(x, y)$ and the only exclusion process for which basic coupling is monotone is the SEP.

The multiple particle jump model

$S = \mathbb{Z}^d$; either $X \subset \mathbb{Z}$ or $X = \mathbb{N}$.
$(\eta_t)_{t \geq 0}$ Markov process on $\Omega = X^S$ with infinitesimal generator :

$$\mathcal{L}f(\eta) = \sum_{x,y \in S} \sum_{\alpha,\beta \in X} \chi_{x,y}^{\alpha,\beta}(\eta) \sum_{k \in \mathbb{N}} \Gamma_{\alpha,\beta}^k (y - x)(f(S_{x,y}^k \eta) - f(\eta))$$

$$\chi_{x,y}^{\alpha,\beta}(\eta) = \begin{cases} 1 & \text{if } \eta(x) = \alpha \text{ and } \eta(y) = \beta \\ 0 & \text{otherwise} \end{cases}$$

$$(S_{x,y}^k \eta)(z) = \begin{cases} \eta(x) - k & \text{if } z = x \text{ and } \eta(x) - k \in X, \eta(y) + k \in X \\ \eta(y) + k & \text{if } z = y \text{ and } \eta(x) - k \in X, \eta(y) + k \in X \\ \eta(z) & \text{otherwise} \end{cases}$$

- assumption : For all $\forall z \in S$, $\alpha, \beta \in X$, $\sum_{k \in \mathbb{N}} \Gamma_{\alpha,\beta}^k (z) < \infty$
- This particle system is conservative : $\eta(x) + \eta(y)$ is a *conserved quantity* in the transition.

Some Classical Examples with $k = 1$:

- SEP : $\Gamma^1_{1,0}(y-x) = p(y-x) \times 1$ [Liggett]
- zero range process (ZRP) : $\Gamma^1_{\alpha,\beta}(y-x) = p(y-x)g(\alpha)$
  $g(\cdot) \nearrow \quad \rightsquigarrow$ attractiveness [Andjel]
  $g(\cdot) \searrow \quad \rightsquigarrow$ condensation [Evans]
- misanthrope process (MP) : $\Gamma^1_{\alpha,\beta}(y-x) = p(y-x)b(\alpha,\beta)$
  misanthropes when $b(\cdot,\cdot) \nearrow$ in α, $\searrow$ in β [Cocozza]
  $\rightsquigarrow$ attractiveness
  philanthropes $\rightsquigarrow$ condensation [Evans]
- target process (TP) : $\Gamma^1_{\alpha,\beta}(y-x) = p(y-x)\mathbf{1}_{\{\alpha\geq 1\}}b(\beta)$ [Godrèche & Luck]

Attractiveness conditions

Theorem

$(\eta_t)_{t\geq 0}$ is attractive iff :
$\forall(\alpha,\beta,\gamma,\delta) \in X^4$ with $(\alpha,\beta) \leq (\gamma,\delta)$, $\forall(x,y) \in S^2$,

$$\forall l \geq 0, \sum_{k'>\delta-\beta+l} \Gamma^{k',}_{\alpha,\beta}(y-x) \leq \sum_{l'>l} \Gamma^{,l'}_{\gamma,\delta}(y-x) \quad (11)$$

$$\forall k \geq 0, \sum_{k'>k} \Gamma^{k',}_{\alpha,\beta}(y-x) \geq \sum_{l'>\gamma-\alpha+k} \Gamma^{,l'}_{\gamma,\delta}(y-x) \quad (12)$$

Non conservative dynamics [Borrello]

Infections+migrations : $S = \mathbb{Z}^d$; $X \subset \mathbb{Z}$ or $X = \mathbb{N}$. $(\eta_t)_{t \geq 0}$
Markov process on $\Omega = X^S$ with infinitesimal generator :

$$\begin{aligned}\mathcal{L}f(\eta) &= \sum_{x,y \in S} p(x,y) \sum_{k>0} \Big(\Gamma^k_{\eta(x),\eta(y)} [f(S^{-k,k}_{x,y}\eta) - f(\eta)] \\ &\quad + (R^{0,k}_{\eta(x),\eta(y)} + P^k_{\eta(y)})[f(S^k_y \eta) - f(\eta)] \\ &\quad + (R^{-k,0}_{\eta(x),\eta(y)} + P^{-k}_{\eta(x)})[f(S^{-k}_x \eta) - f(\eta)] \Big)\end{aligned}$$

denote
$\Pi^{0,k}_{\eta(x),\eta(y)} = R^{0,k}_{\eta(x),\eta(y)} + P^k_{\eta(y)}$, $\Pi^{-k,0}_{\eta(x),\eta(y)} = R^{-k,0}_{\eta(x),\eta(y)} + P^{-k}_{\eta(x)}$

- Attractiveness conditions : inequalities combining infection and migration rates ;
- Increasing coupling : rates that couple infections with migrations.

$\rightsquigarrow$ analysis of ergodicity ; survival or extinction of species, $\cdots$

The coupling rates for $S = \mathbb{Z}$

Proposition

Under conditions (3)–(4)*, the generator* $\overline{\mathcal{L}}$ *with coupling rates* $G_{\xi,\zeta}$ *below, defines an increasing Markovian coupling. Let* $\xi, \zeta \in \Omega$*,* $\forall (x_1, y_1) \in S^2$*,*

$$G_{\xi,\zeta}(x_1, y_1; x_1, y_1) = \Gamma_\xi(x_1, y_1) \wedge \Gamma_\zeta(x_1, y_1). \qquad (13)$$

$\forall (x_1, y_1, x_2, y_2) \in S^4$ *such that* $(x_1, y_1) \neq (x_2, y_2)$*,*
$G_{\xi,\zeta}(x_1, y_1; x_2, y_2) =$

$$\begin{cases} \delta_{x_1,x_2}\left[H^i_{\xi,\zeta}(x_1; y_1, y_2)\right]^+ + \delta_{y_1,y_2}\left[H^f_{\xi,\zeta}(x_1, x_2; y_1)\right]^+ & \text{if } \xi \leq \zeta, \\ \delta_{x_1,x_2}\left[H^i_{\zeta,\xi}(x_1; y_2, y_1)\right]^+ + \delta_{y_1,y_2}\left[H^f_{\zeta,\xi}(x_2, x_1; y_1)\right]^+ & \text{if } \xi > \zeta, \\ 0 & \text{otherwise,} \end{cases} \qquad (14)$$

with, $\forall (x,y,z) \in S^3$,

$$
\begin{aligned}
H^i_{\xi,\zeta}(x;y,z) = \Big(\sum_{y' \le y} (1-\xi(y'))\zeta(y')\Gamma_\xi(x,y')\Big) \\
\wedge \Big(\sum_{z' \le z} (1-\zeta(z'))\big[\Gamma_\zeta(x,z') - \Gamma_\xi(x,z')\big]^+\Big) \\
- \Big(\sum_{y' < y} (1-\xi(y'))\zeta(y')\Gamma_\xi(x,y')\Big) \\
\vee \Big(\sum_{z' < z} (1-\zeta(z'))\big[\Gamma_\zeta(x,z') - \Gamma_\xi(x,z')\big]^+\Big),
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
H^f_{\xi,\zeta}(x,y;z) = \Big(\sum_{x' \le x} \xi(x')\big[\Gamma_\xi(x',z) - \Gamma_\zeta(x',z)\big]^+\Big) \\
\wedge \Big(\sum_{y' \le y} \zeta(y')(1-\xi(y'))\Gamma_\zeta(y',z)\Big) \\
- \Big(\sum_{x' < x} \xi(x')\big[\Gamma_\xi(x',z) - \Gamma_\zeta(x',z)\big]^+\Big) \\
\vee \Big(\sum \zeta(y')(1-\xi(y'))\Gamma_\zeta(y',z)\Big).
\end{aligned}
\tag{16}
$$