Genèse du Livre Jaune

Guy Fayolle * **Roudolf Iasnogorodski** **Vadim Malyshev** †

Un bel exemple de sérendipité ...

Serendipity: « *Don de faire par hasard des découvertes fructueuses* » ou encore « *Faire par hasard une découverte inattendue qui s'avère ensuite fructueuse* ».

▶ Un mot anglais créé par Horace Walpole et qu'il avait tiré d'un conte oriental, *Les Trois Princes de Serendip (1754)*. Serendip serait donc cette terre bénie des dieux où la fortune semble être offerte à chacun...

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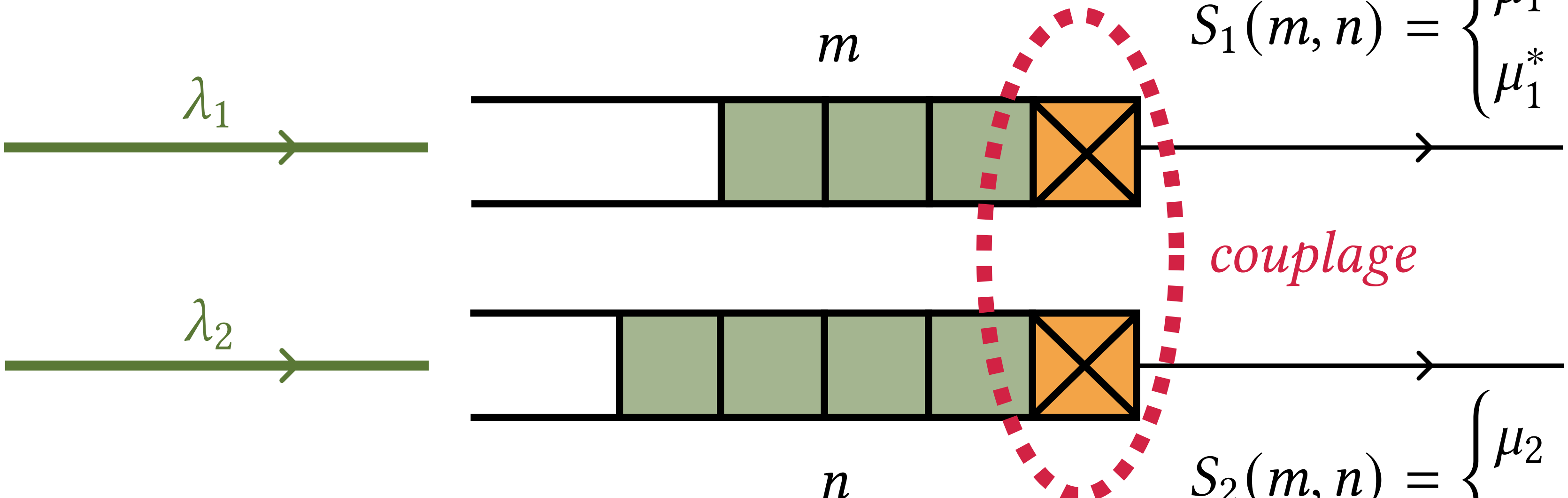
Un petit problème de modélisation que me posa le professeur Edward Coffman en 1975...



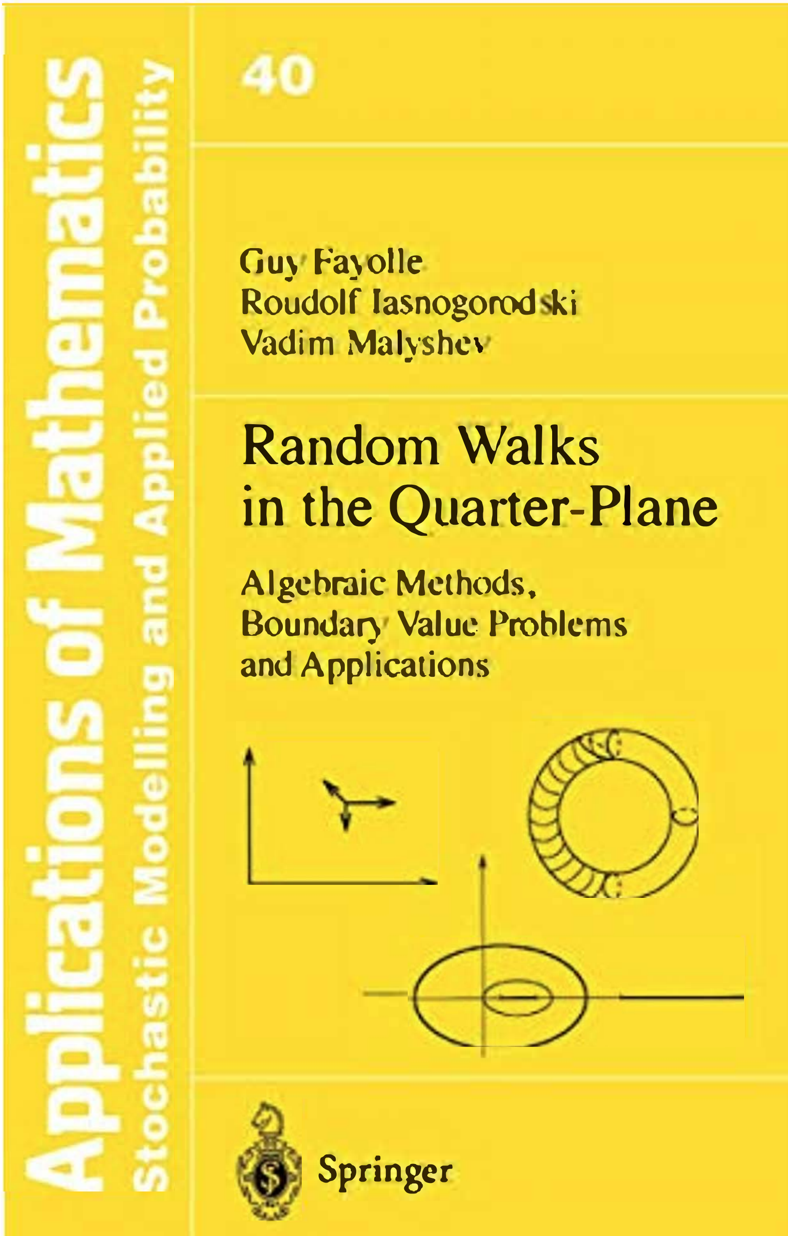
$$S_1(m,n) = \begin{cases} \mu_1 & \text{if } n > 0 \\ \mu_1^* & \text{if } n = 0 \end{cases}$$

$$S_2(m,n) = \begin{cases} \mu_2 & \text{if } m > 0 \\ \mu_2^* & \text{if } m = 0 \end{cases}$$

- Cas particulier : la politique de couplage dite *head of line processor sharing*

$$\mu_1 = \xi \mu_1^*, \quad \mu_2 = (1-\xi)\mu_2^*, \quad 0 \leq \xi \leq 1,$$

... avec le résultat suivant ...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Random Walks
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Algebraic Methods,
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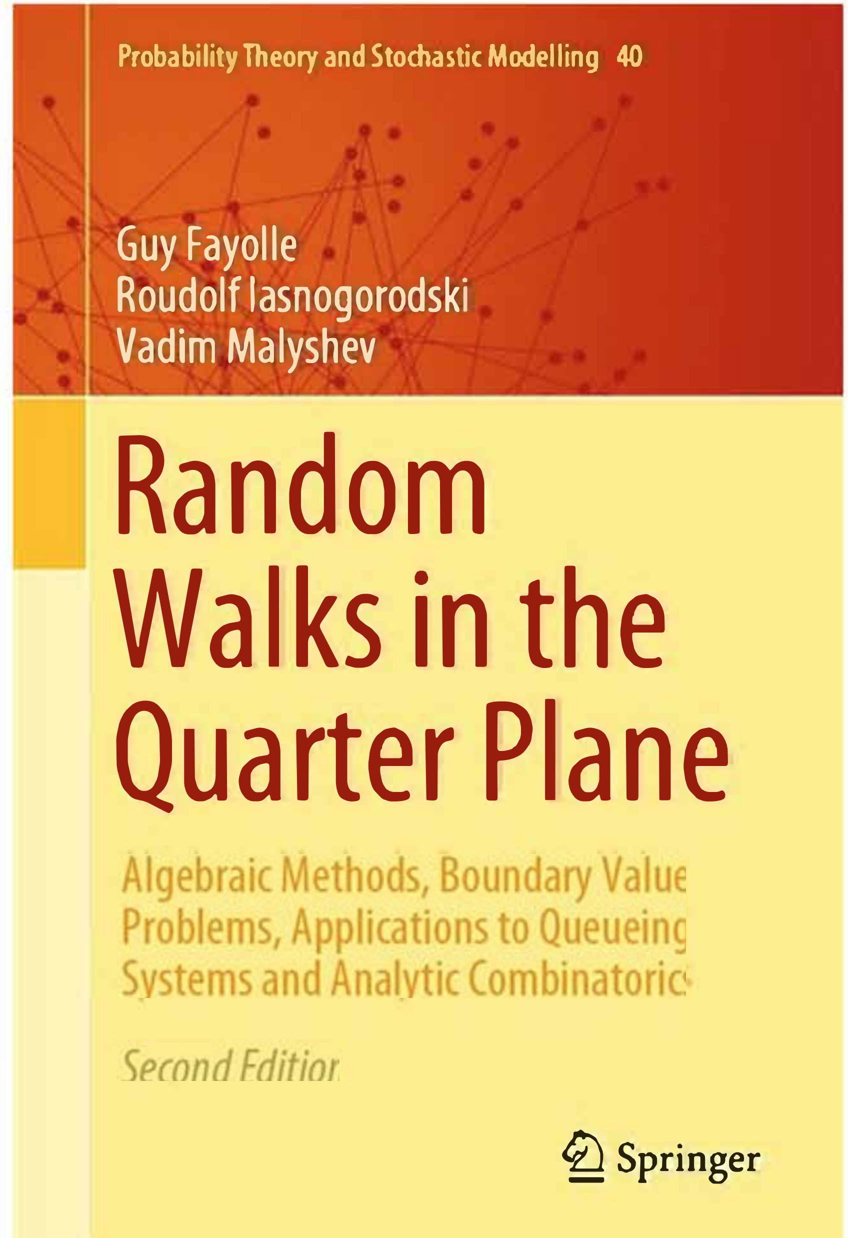
Probability Theory and Stochastic Modelling 40
Guy Fayolle
Roudolf Iasnogorodski
Vadim Malyshev
Random
Walks in the
Quarter Plane
Algebraic Methods, Boundary Value
Problems, Applications to Queueing
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Second Editio
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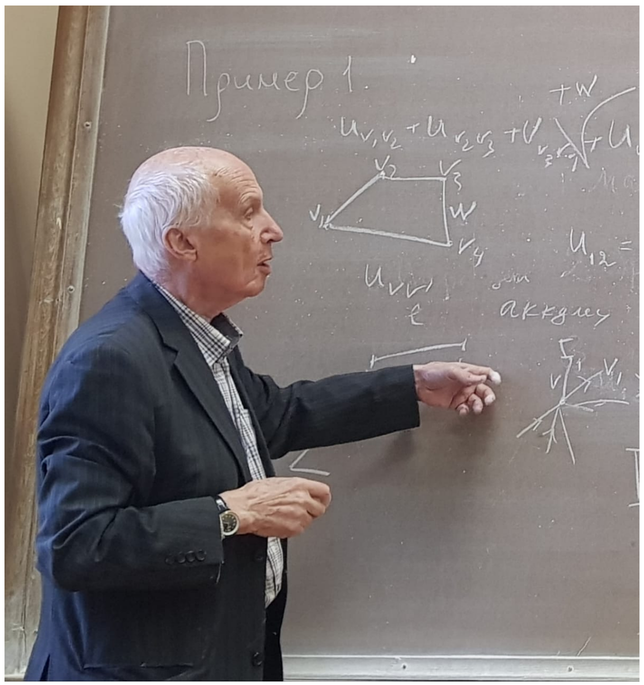
À propos du ≪ Livre jaune ≫ [Roudolf Iasnogorodski]

J'ai rencontré Guy à l'INRIA au milieu des années 1970. Il m'a parlé d'un problème qui l'intéressait beaucoup, concernant deux files d'attente couplées dépendantes. À cette époque, la résolution des réseaux de files à l'état stationnaire reposait sur les résultats de Jackson et des 4 auteurs BCMP, mais ne concernaient que des files indépendantes.

Guy était le seul à travailler sur ce problème. Je me suis alors rappelé avoir eu une discussion avec Vadim Malyshev au début des années 60, lors de mon passage à l'université de Moscou.

Celui-ci travaillait à cette époque sur le problème de deux variables aléatoires dépendantes continues. Cela ne m'a alors pas beaucoup intéressé, car je ne voyais pas d'application directe avec mon travail de l'époque sur la méthode Wiener-Hopf. Mais en parlant avec Guy, qui était très intéressé par les méthodes analytiques et les théorèmes de Spitzer, je me suis souvenu de cette discussion et pensé qu'il pouvait y avoir une relation entre les travaux de Malyshev et les siens. Avec Guy, nous avons donc commencé à chercher des articles de Malyshev sur le sujet, sachant que l'accès aux travaux scientifiques provenant de l'URSS n'était alors pas très simple. Il y en avait quelques-uns et leur étude nous prit un certain temps, car ils étaient difficiles. On y voyait clairement des points communs avec la problématique de Guy, notamment l'approche à l'aide de fonctions génératrices. Cependant, les méthodes que nous avions en vue étaient différentes et, indépendamment, nous avons continué à étudier le problème de Guy, ce qui aboutit (après deux ans de travail intense !) à l'article (maintenant célèbre) publié en 1979 dans le journal probabiliste ZWT (devenu depuis lors PTRF), où était montrée la réduction à un problème aux limites de type Riemann-Hilbert. Notre fructueuse collaboration (scientifique et humaine) s'est poursuivie pendant plus de 50 ans, et a vu la publication du **Livre jaune** et de plusieurs articles concernant les marches aléatoires dans le quart de plan...

Vadim Alexandrovich Malyshev [13/04/1938 – 30/09/2022]



https://www.inria.fr/fr/hommage-vadim-malyshev-specialiste-calcul-probabilites-physique
https://malyshev85.org/

Brief historical summary

- *V. Malyshev*, 1969-1972, at Moscow University. He was the first to be interested in these problems. In particular, in the case of bounded jumps, he answered some questions [asked by A.N. Kolmogorov] about the classification of the walks. He also proposed a solution of (1) via uniformisation.

- *G. Fayolle and R. Iasnogorodski*, 1975-1977, in Paris. Independently, they showed the equivalence to a BVP of Riemann-Hilbert type, yielding solutions in integral form.

- All three have combined their efforts in the so-called **Small Yellow Book** **(**Маленькая Жёлтая Книга**)**. . .

- I also would like to express my gratitude to J.W. Cohen for his continued interest in our analytical approach to random walks problems. His famous book the *Single Server Queue* played an important role in my involvement in stochastic modelling.

Part I: The General Theory

▶ *This part (8 chapters) presents the theoretical foundations and essentially corresponds to the content of the first edition, except for Sections 4.1–4.3 and Chapter 7 which are new. Chapter 8 makes the link with related problems.*

1. Probabilistic Background
2. Foundations of the Analytic Approach

3. Analytic Continuation of the Unknown Functions in the Genus 1 case

4. The Case of a Finite Group

5. Solution in the Case of an Arbitrary Group

6. The Genus 0 Case
7. Explicit Criterion for the Finiteness of the Group in the Genus 0 case

8. Miscellanea: Transient behaviour. About explicit solutions. Asymptotics. Generalized problems and analytic continuation. Outside probability...

Part II: Applications to Queueing Systems and Analytic Combinatorics

9. A Two-Coupled Processor Model

10. Joining the Shorter of Two Queues: Reduction to a Generalized BVP

11. Counting Lattice Walks in the Quarter-Plane

▶ *Chapters* $9 - 11$ *belong to the second edition and borrow specific case-studies from queueing theory and enumerative combinatorics*.

The basic functional equation

The invariant measure $\{\pi_{i,j}, i, j \geq 0\}$ satisfies the fundamental bivariate functional equation

$$\boxed{Q(x,y)\pi(x,y) = q(x,y)\pi(x) + \widetilde{q}(x,y)\widetilde{\pi}(y) + \pi_0(x,y)} \qquad (1)$$

Problem: *Find functions* $\pi(x,y)$, $\pi(x)$, $\widetilde{\pi}(y)$, *satisfying* (1), *holomorphic in* $\mathcal{D} \times \mathcal{D}$ *and continuous in* $\overline{\mathcal{D}} \times \overline{\mathcal{D}}$, *where*

$$\mathcal{D} \stackrel{\text{def}}{=} \{z \in \mathbb{C} : |z| < 1\} \quad \text{and} \quad \overline{\mathcal{D}} \stackrel{\text{def}}{=} \{z \in \mathbb{C} : |z| \leqslant 1\}.$$

- The functions $Q(x,y), q(x,y), \widetilde{q}(x,y)$ are known.
- The function $\pi_0(x,y)$ depends linearly on $L + M - 1$ unknown constants.
- $Q(x,y)$ is often referred to as the kernel of (1).

$$
\begin{cases}
\pi(x,y) = \displaystyle\sum_{i,j\geqslant 1} \pi_{ij} x^{i-1} y^{j-1}, \\
\pi(x) = \displaystyle\sum_{i\geqslant L} \pi_{i0} x^{i-L}, \quad \widetilde{\pi}(y) = \sum_{j\geqslant M} \pi_{0j} y^{j-M}, \\
Q(x,y) = xy\left[1 - \displaystyle\sum_{i,j\in\mathcal{S}} p_{ij} x^i y^j\right], \quad \sum_{i,j\in\mathcal{S}} p_{ij} = 1, \\
q(x,y) = x^L\left[\displaystyle\sum_{i\geqslant -L, j\geqslant 0} p'_{ij} x^i y^j - 1\right] \equiv x^L(P_{L0}(x,y) - 1), \\
\widetilde{q}(x,y) = y^M\left[\displaystyle\sum_{i\geqslant 0, j\geqslant -M} p''_{ij} x^i y^j - 1\right] \equiv y^M(P_{0M}(x,y) - 1), \\
\pi_0(x,y) = \displaystyle\sum_{i=1}^{L-1} \pi_{i0} x^i \left[P_{i0}(x,y) - 1\right] + \sum_{j=1}^{M-1} \pi_{0j} y^j \left[P_{0j}(x,y) - 1\right] + \pi_{00}(P_{00}(xy) - 1).
\end{cases}
$$

$\mathcal{S}$ is the set of allowed jumps, and $q, \widetilde{q}, q_0, P_{i0}, P_{0j}$, are given probability generating functions supposed to have suitable analytic continuations (as a rule, they are polynomials when the jumps are bounded).

The new approaches to the solution proposed in the Yellow Book are going far beyond simply obtaining an index theory for the quarter-plane...

1. The first step, quite similar to a Wiener–Hopf factorization, consists in considering the above equation on the algebraic curve $\mathcal{A} = \{(x, y) \in \mathbb{C}^2 : Q(x, y) = 0\}$. (which is elliptic in the generic situation), so that we are then left with an equation for two unknown functions of one variable on this curve.

2. Next a crucial idea is to use Galois automorphisms on this algebraic curve. More information is obtained by using the fact that the unknown functions π and $\widetilde{\pi}$ depend solely on x and y respectively.

   It is possible to prove that π and $\widetilde{\pi}$ can be lifted as meromorphic functions onto the universal covering of some Riemann surface $\mathbf{S}$. Here $\mathbf{S}$ corresponds to the algebraic curve $\mathcal{A}$. When $g \stackrel{\text{def}}{=}$ the genus of $\mathbf{S}$ is 1 [Torus] (resp. 0), the universal covering is the complex plane $\mathbb{C}$ (resp. the Riemann sphere).

3. Lifted onto the universal covering, π (and also $\widetilde{\pi}$) satisfies a system of non-local equations having the simple form

$$\begin{cases} \pi(t+\omega_1) = \pi(t), \quad \forall t \in \mathbb{C}\,, \\ \pi(t+\omega_3) = a(t)\pi(t) + b(t), \quad \forall t \in \mathbb{C}\,, \end{cases}$$

where ω_1 [resp. ω_3] is a complex [resp. real] constant. The solution can be presented in terms of infinite series equivalent to Abelian integrals. The backward transformation (projection) from the universal covering onto the initial coordinates can be given in terms of uniformization functions, which for $g = 1$ are Weierstrass $\wp$ elliptic functions.

4. Another direct approach consists in working solely in the complex plane $\mathbb{C}$. After analytic continuation, it appears that the determination of π reduces to a boundary value problem (BVP), belonging to the Riemann–Hilbert–Carleman class, the basic form of which can be formulated as follows.

▶ $\mathscr{G}(\mathcal{L}) \stackrel{\text{def}}{=}$ interior of the domain bounded by a simple smooth closed contour $\mathcal{L}$.

Find a function Φ^+ holomorphic in $\mathscr{G}(\mathcal{L})$, the limiting values of which are continuous on the contour and satisfy the relation

$$\boxed{\Phi^+(\alpha(t)) = G(t)\Phi^+(t) + g(t), \quad t \in \mathcal{L}.} \tag{2}$$

▶ g, $G \in \mathbb{H}_\mu(\mathcal{L})$ (*Hölder condition* with parameter μ on $\mathcal{L}$).

▶ α, the shift, is a function establishing a one-to-one mapping of the contour $\mathcal{L}$ onto itself, such that the direction of traversing $\mathcal{L}$ is changed and

$$\alpha'(t) = \frac{d\alpha(t)}{dt} \in \mathbb{H}_\mu(\mathcal{L}), \quad \alpha'(t) \neq 0, \quad \forall t \in \mathcal{L}.$$

▶ α is most frequently subject to the so-called Carleman condition

$$\alpha(\alpha(t)) = t, \quad \forall t \in \mathcal{L}, \quad \text{where typically} \quad \alpha(t) = \bar{t}.$$

The point of this method resides in the fact that solutions are given in terms of explicit integral-forms.

Solving (1) by reduction to a BVP

To find functions $\pi(x, y)$, $\pi(x)$, $\widetilde{\pi}(y)$, satisfying (1), holomorphic in $\mathcal{D} \times \mathcal{D}$ and continuous in $\overline{\mathcal{D}} \times \overline{\mathcal{D}}$, 3 steps are proposed.

Sketch of the 3 main steps.

- Restrict (1) to the algebraic curve $\mathcal{A} = \{(x, y) \in \mathbb{C}^2 : Q(x, y) = 0\}$.

  When Q is irreducible, $\mathcal{A}$ is associated with a compact Riemann surface of genus g, where $g = 1$ [Torus] or $g = 0$ [Sphere].

- Eliminate one function, say $\widetilde{\pi}(y)$.

- Formulate a Boundary Value Problem for $\pi(x)$.

Step 1 [Restriction to $\mathcal{A} = \{(x,y) \in \mathbb{C}^2 : Q(x,y) = 0\}$, assuming $g = 1$]

Q(x,y) is a polynomial of degree 2 w.r.t. each variable, and of degree 4 in (x,y).

$$\boxed{Q(x,y) = a(x)y^2 + b(x)y + c(x) = \widetilde{a}(y)x^2 + \widetilde{b}(y)x + \widetilde{c}(y)}, \tag{3}$$

Let the discriminants of the kernel, in the respective $\mathbb{C}_x$ and $\mathbb{C}_y$ complex planes,

$$D(x) = b(x)^2 - 4a(x)c(x), \qquad \widetilde{D}(y) = \widetilde{b}(y)^2 - 4\widetilde{a}(y)\widetilde{c}(y). \tag{4}$$

Then $D(x)$ [resp. $\widetilde{D}(y)$] has 4 real roots in $\mathbb{C}_x$ (resp. $\mathbb{C}_y$) satisfying

$$|x_1| < x_2 < 1 < x_3 < |x_4| \leqslant \infty,$$
$$|y_1| < y_2 < 1 < y_3 < |y_4| \leqslant \infty,$$

Moreover, the slits $[x_1, x_2]$ and $[y_1, y_2]$ are included in the segment $[-1, +1]$.

Let the algebraic functions $X(y)$ and $Y(x)$ be defined by

$$\boxed{Q(X(y), y) = Q(x, Y(x)) = 0}.$$

- $X(\cdot)$ has two branches $X_0(y), X_1(y)$, which are meromorphic in $\mathbb{C}_y$ cut along $[y_1, y_2] \cup [y_3, y_4]$.
  - These two branches can be separated to ensure $|X_0| \leqslant |X_1|, \ \forall y \in \mathbb{C}_y$.
  - On the cut $[y_1, y_2] \cup [y_3, y_4]$, we have $X_0 = \overline{X}_1$ (complex conjugate).
- On the circle $|y| = 1$, we have $|X_0(y)| \leqslant 1$.

Similarly, $Y_0(x), Y_1(x)$ are the two branches of $Y(\cdot)$ defined in $\mathbb{C}_x$ with the respective properties. . .

Step 2 [Elimination of $\widetilde{\pi}(y)$]

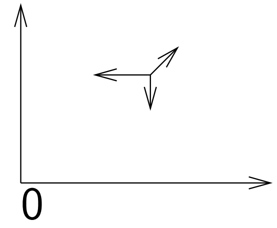
▶ Starting from the region $\mathcal{A} \cap (\overline{\mathcal{D}} \times \overline{\mathcal{D}})$, the study of the branches allows to continue $\pi(x)$ and $\widetilde{\pi}(y)$ as meromorphic functions to the whole complex plane, cut respectively along $[x_3, x_4]$ and $[y_3, y_4]$.

▶ When y tends successively from above (y^+) and from below (y^-) to an arbitrary point y of the cut $[y_1, y_2]$, $\widetilde{\pi}(y)$ remains continuous, and thus can be eliminated by computing the difference $\widetilde{\pi}(y^+) - \widetilde{\pi}(y^-) = 0$.

▶ Let $[\overrightarrow{\underleftarrow{y_1, y_2}}]$ stand for a contour representing the slit $[y_1, y_2)]$ traversed from y_1 to y_2 along the upper edge, and then back to y_1 along the lower edge. The object

$$\mathcal{M} = X_0([\overrightarrow{\underleftarrow{y_1, y_2}}]) = \overline{X}_1([\overleftarrow{\underrightarrow{y_1, y_2}}])$$

is a smooth closed contour in $\mathbb{C}_x$, which is part of a quartic curve when $g = 1$.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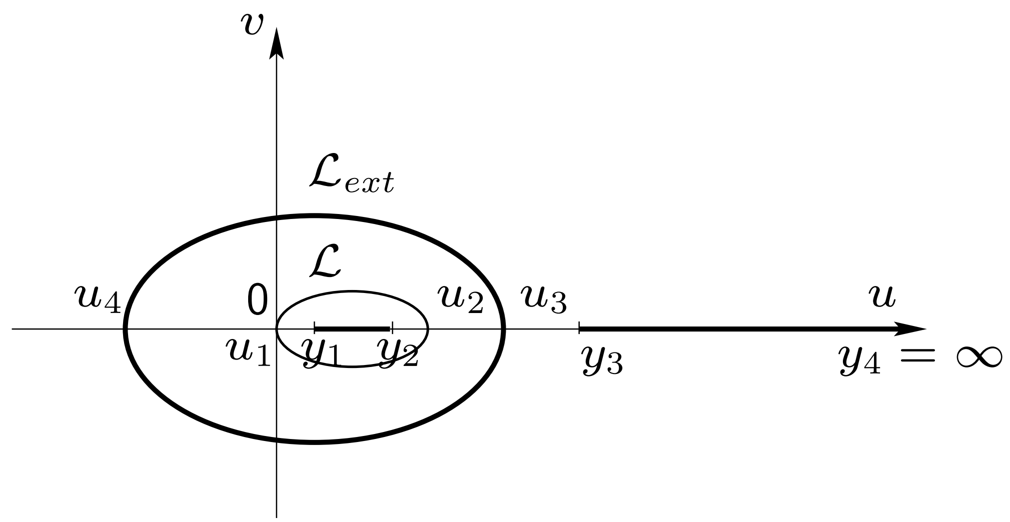
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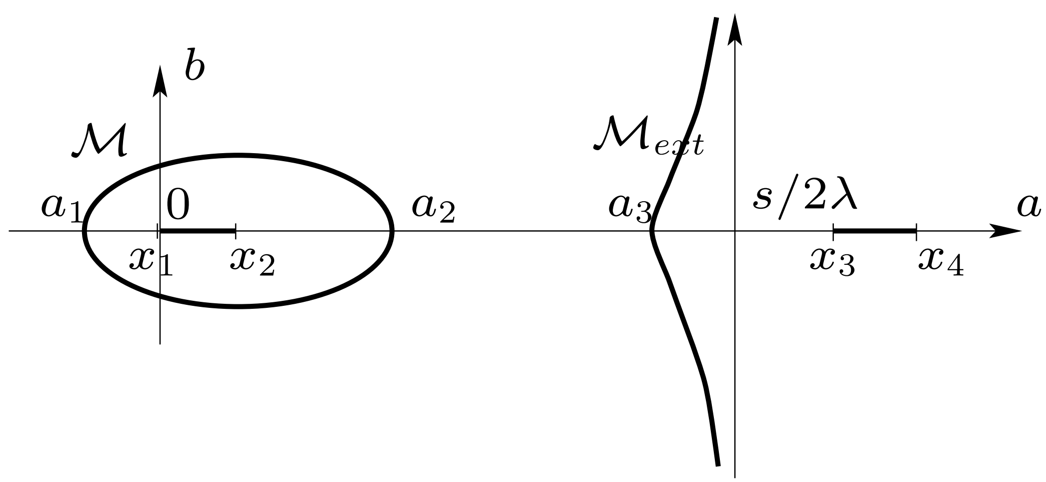
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Step 3 [Formulation of the Riemann-Hilbert-Carleman BVP with a shift]

From the argument used in **Step 2**, we can write

$$\boxed{\pi(t)A(t) - \pi(\bar{t})A(\bar{t}) = g(t), \quad t \in \mathcal{M}},$$

which is a problem of type (2) for the domain $\mathscr{G}(\mathcal{M})$ bounded by the closed contour $\mathcal{M}$, the function $\pi(x)$ being meromorphic (at most one pole) in $\mathscr{G}(\mathcal{M})$.

Theorem 1. *Under the ergodicity condition* (6), *the function* π *is given by*

$$\pi(x) = \frac{U(x)H(x)}{2i\pi} \int_{\mathcal{M}_d} \frac{K(t)w'(t)dt}{H^+(t)(w(t) - w(x))} + V(x), \ \forall x \in \mathscr{G}(\mathcal{M}). \quad (5)$$

1. $\mathcal{M}_d$ *is the portion of the curve* $\mathcal{M}$ *located in the lower half-plane* $\Im z \leqslant 0$.
2. U, V, K *are known functions, all involving some specific zeros of* $\widetilde{q}(X_0(y), y)$ *and* $q(x, Y_0(x))$ *inside* $\mathscr{G}(\mathcal{M})$*; moreover* U, V *are rational fractions.*
3. w *is a* gluing function, *which realizes the* conformal mapping *of* $\mathscr{G}(\mathcal{M})$ *onto the complex plane cut along a segment and has an explicit form via the Weierstrass* $\wp$*-function.*

Theorem 2. *Introduce the following two quantities:*

$$\delta \stackrel{def}{=} \begin{cases} 0, \text{ if } Y_0(1) < 1 \text{ or } \left\{ Y_0(1) = 1 \text{ and } \dfrac{dq(x, Y_0(x))}{dx}_{|x=1} > 0 \right\}, \\ 1, \text{ if } Y_0(1) = 1 \text{ and } \dfrac{dq(x, Y_0(x))}{dx}_{|x=1} < 0. \end{cases}$$

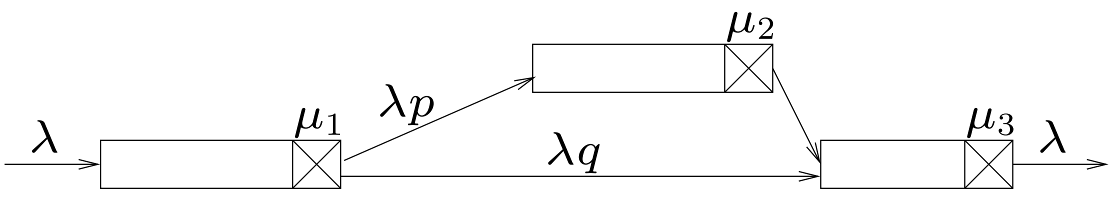
$$\widetilde{\delta} \stackrel{def}{=} \begin{cases} 0, \text{ if } X_0(1) < 1 \text{ or } \left\{ X_0(1) = 1 \text{ and } \dfrac{d\widetilde{q}(X_0(y), y)}{dy}_{|y=1} > 0 \right\}, \\ 1, \text{ if } X_0(1) = 1 \text{ and } \dfrac{d\widetilde{q}(X_0(y), y)}{dy}_{|y=1} < 0. \end{cases}$$

Then (1) *admits a probabilistic solution if, and only if,*

$$\delta + \widetilde{\delta} = \mathbb{1}_{\{X_0(1)=1, Y_0(1)=1\}} + 1, \tag{6}$$

which are the necessary and sufficient conditions for the random walk to be ergodic. ■

Example 1: Sojourn time in a 3-node Jackson network with overtaking



- An inherent overtaking phenomenon renders things slightly complicated.
- Cutting the Gordian Knot amounts to finding the function

$G(x, y, z, s) \stackrel{\text{def}}{=}$ Laplace transform of the conditional waiting time distribution of a tagged customer at a departure instant of the first queue...

The following non-homogeneous FE holds, for $|x|, |y|, |z| \leq 1, \Re(s) \geq 0$,

$$K(x,y,z,s)G(x,y,z,s) = \left(\mu_1 - \frac{\lambda}{x}\right) G(0,y,z,s) + \left(\mu_3 - q\mu_1\frac{x}{z} - \mu_2\frac{y}{z}\right) G(x,y,0,s) + \frac{\mu_2\mu_3}{(1-x)[s+\mu_3(1-z)]},$$

where p, q are routing probabilities with $p + q = 1$, λ is the external arrival rate, μ_i is the service rate at queue i, and

$$K(x,y,z,s) = s + \lambda\left(1 - \frac{1}{x}\right) + \mu_1\left(1 - px - q\frac{x}{z}\right) + \mu_2\left(1 - \frac{y}{z}\right) + \mu_3(1-z).$$

- Considering y and s as parameters, the above FE becomes

$$\boxed{K(x,y,z,s)\widetilde{G}(x,z) = A(x)\widetilde{G}(0,z) + B(x,y,z,s)\widetilde{G}(x,0) + C(x,y,z,s)}.$$

Finally, the LST of the total sojourn time of an arbitrary customer is given by

$$(1-\rho_1)(1-\rho_2)(1-\rho_3)\frac{\mu_1}{\mu_1+s}G\left[\frac{\mu_1}{\mu_1+s}, \rho_2, \rho_3, s\right].$$

Example 2: Counting lattice walks in the quarter plane $\mathbb{Z}_+^2$

For a given set $\mathcal{S}$ of admissible steps (or jumps), it is a matter of counting the number of paths of a certain length, which start and end at some arbitrary points, and might even be restricted to some region of the plane.

Three natural questions arise.

- Q1: How many such paths do exist?

- Q2: What is the asymptotic behavior, as their length goes to infinity, of the number of walks ending at some given point or domain (for instance one axis)?

- Q3: What is the nature of the generating function of the numbers of walks (*rational*, *algebraic*, *holonomic*)? If paths remain in a half-plane, then the CGFs have explicit forms and can only be rational or algebraic (see e.g., [BM-P-2003]). The situation happens to be much richer if the walks are confined to $\mathbb{Z}_+^2$...

Questions Q1, Q2 for walks in $\mathbb{Z}_+^2$ with small steps.

▶ Walks start at the origin.

▶ The set $\mathcal{S}$ of admissible steps is included in the set of the 8 nearest neighbors, i.e., $\mathcal{S} \subset \{-1,0,1\}^2 \setminus \{(0,0)\}$. By using an extended Kronecker's delta, we shall write

$$\delta_{i,j} = \begin{cases} 1 & \text{if} \quad (i,j) \in \mathcal{S}, \\ 0 & \text{if} \quad (i,j) \notin \mathcal{S}. \end{cases}$$

▶ Walks are supposed to be continuous on the boundary of $\mathbb{Z}_+^2$: allowed jumps are the natural ones and jumps that would take the walk out of $\mathbb{Z}_+^2$ are discarded.

A priori, there are 2^8 such models. But, by using simple geometrical symmetries [BM-M 2010], there are in fact only 79 types of essentially distinct walks.

Let $f(i,j,k) \stackrel{\text{def}}{=}$ the number of paths in $\mathbb{Z}_+^2$ starting from $(0,0)$ and ending at (i,j) after k steps. Then

$$F(x,y,z) \stackrel{\text{def}}{=} \sum_{i,j,k\geqslant 0} f(i,j,k) x^i y^j z^k \tag{7}$$

satisfies the functional equation

$$\boxed{K(x,y,z)F(x,y,z) = c(x)F(x,0,z) + \widetilde{c}(y)F(0,y,z) + c_0(x,y,z)}, \tag{8}$$

a priori valid in the domain $|x| \leqslant 1, |y| \leqslant 1, |z| < 1/|\mathcal{S}|$, where

$$\begin{cases} K(x,y;z) = xy\left[\displaystyle\sum_{(i,j)\in\mathcal{S}} x^i y^j - 1/z\right], \\ c(x) = \displaystyle\sum_{i\leqslant 1} \delta_{i,-1} x^{i+1}, \quad \widetilde{c}(y) = \displaystyle\sum_{j\leqslant 1} \delta_{-1,j} y^{j+1}, \\ c_0(x,y,z) = -\delta_{-1,-1} F(0,0,z) - xy/z, \end{cases}$$

Proposition 3. *For $x \in \mathscr{G}(\mathcal{M}_z)$,*

$$\boxed{c(x)F(x,0,z) - c(0)F(0,0,z) = \frac{1}{2\pi i z}\int_{\mathcal{M}_z} tY_0(t,z)\frac{w'(t,z)}{w(t,z) - w(x,z)}\,\mathrm{d}t}, \quad (9)$$

where $w(x,z)$ is the gluing function for the domain $\mathscr{G}(\mathcal{M}_z)$ in the $\mathbb{C}_x$-plane. A similar expression can be written for $F(0,y,z)$.

By symmetry and classical arguments, only the real singularities of $F(0,0,z)$, $F(1,0,z)$, $F(0,1,z)$ and $F(1,1,z)$ with respect to z will play a role in the asymptotics. From the expression (9), the main source of all possible singularities can be explained. We simply quote the main result (see [F-R 2012]).

Proposition 4. *The smallest positive singularity of $F(0,0,z)$ is*

$$\boxed{z_g = \inf\{z > 0 : y_2(z) = y_3(z)\}}. \quad (10)$$

Questions Q3 [nature of the functions ?]

▶ Let $\mathbb{C}(x), \mathbb{C}(y)$ and $\mathbb{C}(x,y)$ denote the respective fields of rational functions of x, y and (x,y) over $\mathbb{C}$. In general Q is assumed to be irreducible, and the quotient field $\mathbb{C}(x,y)$ with respect to Q will be denoted by $\mathbb{C}_Q(x,y)$.

Definition 5. The group of the random walk is the Galois group $\mathcal{H} = \langle \xi, \eta \rangle$ of automorphisms of $\mathbb{C}_Q(x,y)$ generated by ξ and η given by

$$\xi(x,y) = \left(x, \frac{c(x)}{y\,a(x)}\right), \qquad \eta(x,y) = \left(\frac{\widetilde{c}(y)}{x\,\widetilde{a}(y)}, y\right).$$

ξ and η are involutions satisfying $\xi^2 = \eta^2 = I$.

Lemma 6. *Let*

$$\delta \stackrel{def}{=} \eta\xi. \tag{11}$$

$\mathcal{H}$ has a normal cyclic subgroup $\mathcal{H}_0 = \{\delta^n, n \in \mathbb{Z}\}$, which is finite or infinite, and $\mathcal{H}/\mathcal{H}_0$ is a cyclic group of order 2. The group $\mathcal{H}$ is finite of order $2n$ if and only if $\delta^n = I$.

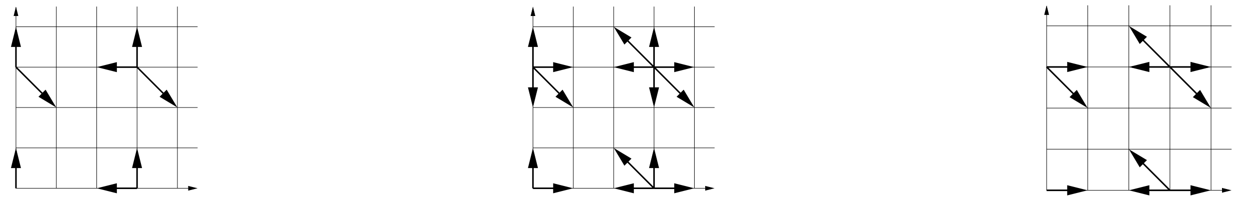


Figure 1: Left, 2 walks with group 6. Right, 1 walk with group 8.

Figure 2: Left, 3 walks with group 6. Right, 1 walk with group 8.

Theorem 7 (BM-M 2010). *For the 16 walks with a group of order 4 and for the 3 walks in figure 1, the formal trivariate series* (7) *is holonomic non-algebraic. For the 3 walks on the left in figure 2, the trivariate series* (7) *is algebraic.*

Theorem 8 (see B-K 2010). *For the so-called* Gessel's walk *on the right in figure 2, the formal trivariate series* (7) *is algebraic.*

They use a less general group acting only on $\mathbb{C}(x, y)$! In [F-R 2010], a direct proof with $\mathcal{H}$, is provided, together with the fact that $\pi(x)$ in equation (1) is always holonomic when the groupe is finite.

About generalizations...

- Arbitrary finite jumps of size n. In this case, the crucial first step is (as before) the analytic continuation process. A priori $2n$ unknown functions of one variable. A functional equation can be obtained on a Riemann surface $\mathbf{S}$ of arbitrary genus, implying the manipulation of hyperelliptic curves.
  The ultimate goal might be to set a generalized BVP on a single curve for a vector of analytic functions: this remains a doable challenge, see [F-R 2015].

- Space inhomogeneity. Here, one must frequently deal with systems of functional equations (see e.g., the example JSQ).

- Larger dimensions. In $\mathbb{Z}_+^n, n \geq 3$, most of the questions are largely open.
  For a first step in this direction, see [Ovseevich-1995], where explicit integral formulas for the resolvent of the discrete Laplace operator in an orthant are given.

  Yet, a global solution to the essential problems seems out of reach, even computationally : analytic continuation, index calculation, BVP for n complex variables...

Aux passionné.es des ballades dans le quart de plan...

D'abord, un profonde reconnaissance à Sandro Franceschi et Kilian Raschel pour leur initiative et l'organisation de cette célébration du 25^e anniversaire du petit Livre Jaune...

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- Et aussi à celles et ceux que je ne connais pas...
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