Strong differential transcendence, difference equations and combinatorics

Un quart de siècle pour un quart de plan, Marseille

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April 15th, 2025

Klazar's theorem

Bell numbers, $B_n :=$ number of partitions of a set of cardinality $n \geq 1$

defined by $\sum_{n\geq 0} \frac{B_n}{n!} t^n := \exp(e^t - 1)$ (EGF)

$B(t) := 1 + \sum_{n\geq 1} B_n t^n = 1 + t + 2t + 5t^3 + 15t^4 + 52t^5 + 203t^6 + 877t^6 + 4140t^7 + 21147t^8 + 115975t^9 + \cdots \in \mathbb{Z}[[t]]$ (OGF)

$$B\left(\frac{t}{1+t}\right) = tB(t) + 1$$

Theorem (Klazar 2003)

$B(t)$ is differentially transcendental over the field of meromorphic functions at 0,
i.e. is not solution of an algebraic differential equation with coefficients meromophic at 0.

RMK. $z(t) := \Gamma\left(\frac{1}{t}\right)^{-1}$ solution of $z\left(\frac{t}{t+1}\right) = tz(t)$

$\rightsquigarrow$ homogeneous eq. associated to $B\left(\frac{t}{t+1}\right) = tB(t) + 1$

Borel transform and EGF

$$\tau(f(t)) = f\left(\frac{t}{1+t}\right)$$

$\hat{\cdot}, \Phi_\tau : \mathbb{C}[[t]] \to \mathbb{C}[[t]]$:

$\left(\sum_{n\geq 0} g_n t^n\right)\hat{} := \sum_{n\geq 0} g_n \frac{t^n}{n!}$ $\qquad$ $\Phi_\tau(f) := \frac{1}{1-t} \cdot f\left(\frac{t}{1-t}\right)$

$\forall f, g \in C[[t]]$:

$$\Phi_\tau(f) = g \Leftrightarrow \hat{g} = \hat{f} \cdot e^t;$$

$$\frac{d}{dt}(\hat{f}) = \left(\frac{f(t)-f(0)}{t}\right)\hat{}.$$

Proposition (Bostan, D.V., Raschel)

"If an EGF is defined by a *closed exponential form*, then the OGF is solution of a linear τ-equation"

Example

$B(t)$ OGF of Bell numbers

$\hat{B}(t) := \exp(\mathrm{e}^t - 1)$ EGF

$\Rightarrow \frac{d}{dt}(\hat{B}) - e^t \cdot \hat{B} = 0$

$\Rightarrow \frac{B-1}{t} = \Phi_\tau(B)$

$\Rightarrow \tau(B) = tB + 1$

Some other examples

polynomial $P_n(x)$	EGF $\sum_{n\geq 0} P_n(x)\frac{t^n}{n!}$	a	b
Bernoulli $B_n(x)$	$\frac{t}{e^t-1}\cdot\exp(xt)$	$1+t$	$-\frac{(1+t)t}{(xt-t-1)^2}$
Glaisher $U_n(x)$	$\frac{t}{e^t+1}\cdot\exp(xt)$	$1+t$	$\frac{(1+t)t}{(xt-t-1)^2}$
Apostol-Bernoulli $A_n^{(\gamma)}(x)$	$\frac{t}{\gamma\, e^t-1}\cdot\exp(xt)$	$\gamma\,(1+t)$	$-\frac{(1+t)t}{(xt-t-1)^2}$
Imschenetsky $S_n(x)$	$\frac{t}{e^t-1}\cdot(\exp(xt)-1)$	$1+t$	$\frac{t^2x(xt-2\,t-2)}{(1+t)(xt-t-1)^2}$
Euler $E_n(x)$	$\frac{2}{e^t+1}\cdot\exp(xt)$	$-(1+t)$	$\frac{2(1+t)}{1+t-xt}$
Genocchi $G_n(x)$	$\frac{2\,t}{e^t+1}\cdot\exp(xt)$	$-(1+t)$	$\frac{2\,(1+t)t}{(1+t-xt)^2}$
Carlitz $C_n^{(\gamma)}(x)$	$\frac{1-\gamma}{1-\gamma\, e^t}\cdot\exp(xt)$	$\gamma\,(1+t)$	$\frac{(1-\gamma)(1+t)}{1+t-xt}$
Fubini $F_n(x)$	$1/(1-x(e^t-1))$	$\frac{x}{x+1}\cdot(1+t)$	$\frac{1}{x+1}$
Bell-Touchard $\phi_n(x)$	$\exp\left(x(e^t-1)\right)$	xt	1
Mahler $s_n(x)$	$\exp\left(x(1+t-e^t)\right)$	$\frac{x(1+t)t}{xt-t-1}$	$\frac{1+t}{1+t-xt}$
Toscano's actuarial $a_n^{(\gamma)}(x)$	$\exp\left(-xe^t+\gamma t+x\right)$	$\frac{x(1+t)t}{\gamma t-t-1}$	$\frac{1+t}{1+t-\gamma t}$

$$\rightsquigarrow \tau(y) = ay + b$$

Klazar's theorem is an instance of a very general phenomenon!

Theorem (Bostan, D.V.,Raschel). Let $a, b \in \mathbb{C}(t)$, with $a \neq 0$, and let $w \in \mathbb{C}((t)) \setminus \mathbb{C}(t)$ verify the difference equation $w\left(\frac{t}{1+t}\right) = aw + b$. Then w is D-transcendental over $\mathbb{C}(\{t\})$.

The proof relies on difference Galois theory.... Franke (1963), van der Put-Singer (1997)

Corollary. All the series in the previous table are D-transcendental over $\mathbb{C}(\{t\})$.

Genus 0 walks in the quarter plane

Walks in $\mathbb{N}^2$ starting at $(0,0)$, with steps $\mathcal{S} \subset \{\rightarrow, \leftarrow, \uparrow, \downarrow, \searrow, \swarrow, \nearrow, \nwarrow\}$

$q_{\mathcal{S}}(i,j;n) :=$ # walks of length n ending at (i,j)

$$Q_{\mathcal{S}}(x,y;t) = \sum_{i,j,n=0}^{\infty} q_{\mathcal{S}}(i,j;n) x^i y^j t^n \in \mathbb{Z}[[x,y,t]]$$

$$\rightsquigarrow K(x,y,t) := xy - t \cdot \textstyle\sum_{(i,j)\in\mathcal{S}} x^{i+1} y^{j+1}$$

$$K(x,y,t)Q(x,y;t) = xy - tx^2 Q(x,0;t) - ty^2 Q(0,y;t)$$

$\rightsquigarrow$ the genus of the curve K is zero,

$\rightsquigarrow$ generically, it admits a rational parametrization

1. $\forall t \in (0, 1/2]$
$\exists (x_t(s), y_t(s)), (\widetilde{x}_t(s), y_t(s)) \in \mathbb{C}(s)^2$:
$K(x_t(s), y_t(s), t) = K(\widetilde{x}_t(s), y_t(s), t) = 0$

$\rightsquigarrow tx_t^2 Q(x_t, 0) - t\tilde{x}_t^{\,2} Q(\tilde{x}_t, 0) = (x_t - \tilde{x}_t) y_t$

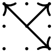
2. $\exists \tau$ homography s.t. $x_t(\tau(s)) = \widetilde{x}_t(s)$
$G(s) := tx_t(s)^2 Q(x_t(s), 0)$ satisfies:

   $\rightsquigarrow G(s) - G(\tau(s)) = (x_t - \tilde{x}_t) y_t \in \mathbb{C}(s)$

3. if τ may have 1 or 2 fixed points...

Ideas coming from [Fayolle, Iasnogorodski, Malyshev, 1999, 2017]

Model $\mathcal{A}$

 $\rightsquigarrow K(x,y) = xy - t\left(y^2 + x^2y^2 + x^2\right)$, with $t = \frac{v}{1+v^2}$ $\quad$ $t \in (0, 1/2) \Leftrightarrow v \in (0, 1)$

$$x_0(s) = \frac{\left(1 - v^2\right) s}{v\left(s^2+1\right)}, \quad y_0(s) = \frac{\left(1 - v^2\right) s}{v^2 s^2 + 1}$$

$$\widetilde{x_0}(s) = \frac{\left(1 - v^2\right) v\, s}{v^4 s^2 + 1}, \text{ with } \widetilde{x_0}(s) = x_0(v^2 s)$$

For $v \in (0, 1)$: $G_0(v^2 s) - G_0(s) = \dfrac{(v^2 - 1)}{v}\left(\dfrac{1}{s^2+1} - \dfrac{2}{v^2s^2+1} + \dfrac{1}{v^4s^2+1}\right)$

$\widetilde{G}_0(s) = \frac{v}{2(v^2-1)} G_0(s) - \frac{1}{2(s^2+1)}$, so that: $\widetilde{G}_0(v^2 s) - \widetilde{G}_0(s) = \frac{1}{s^2+1} - \frac{1}{v^2s^2+1}$.

Of course, $G_0(s)$ is rational if and only if $\widetilde{G}_0(s)$ is rational.

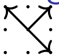
Theorem (Bostan-DV-Rascel)

$Q(x, 0, t)$ (resp. $Q(0, y, t)$, $Q(x, y, t)$) is D-transc.$/\mathbb{C}(x)$ (resp. $\mathbb{C}(y)$, $\mathbb{C}(x, y)$), for any $t \in (0, 1/2)$.

Hint. $G_0(s)$ is either rational or diff. transcendental over $\mathbb{C}(s)$ by Ishizaki-Ogawara Theorem (1998-2015).

Galoisian statements in this framework [Dreyfus,Hardouin, Roques, Singer], for $t \in (0, 1/\#\mathcal{S})$ transcendent, for models with weights.

Model $\mathcal{A}$ at the sigular point



$$\rightsquigarrow K(x,y) = xy - \tfrac{1}{2}\left(y^2 + x^2y^2 + x^2\right) \qquad \boxed{t = 1/2 \Leftrightarrow v = 1}$$

$$K(x,y) = \frac{1}{2}\,(ix - iy + xy)\,(ix - iy - xy) \Rightarrow \left(x, y(x) := \frac{ix}{1+ix}\right) \in \mathbb{C}(x)^2$$

$G_1(x) := \frac{x^2}{2} Q\left(x, 0, \frac{1}{2}\right)$ verifies the functional equation

$$G_1\left(\tau(x)\right) = G_1(x) + \frac{ix^2}{1+ix}, \text{ with } \tau(x) = \tfrac{ix}{1+ix}.$$

Theorem (Bostan-DV-Raschel)

1. $Q(x, 0, \frac{1}{2}) \in \mathbb{Q}[[x,y]]$;
2. G_1 is D-transc.$/\mathbb{C}(\{x\})$;
3. $Q(x, 0, \frac{1}{2}) = \sum_{n\geq 0} 2(2^{2n+2} - 1)\frac{(-1)^n}{n+1} B_{2n+2}x^{2n}$,
   where $\sum_{n\geq 0} B_n \frac{x^n}{n!} = \frac{x}{e^x - 1}$ (Bernoulli numbers $(B_n)_{n\geq 0}$).

- $Q(x, y, 1/2)$ is in $\mathbb{Q}[[x,y]]$ (proof inspired by [Mishna, Rechnitzer])
- Explicit expression with Bernoulli numbers for model $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{D}$.
- $t = \frac{v}{1+v^2}$ [Mishna, Rechnitzer]

Thank you!

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