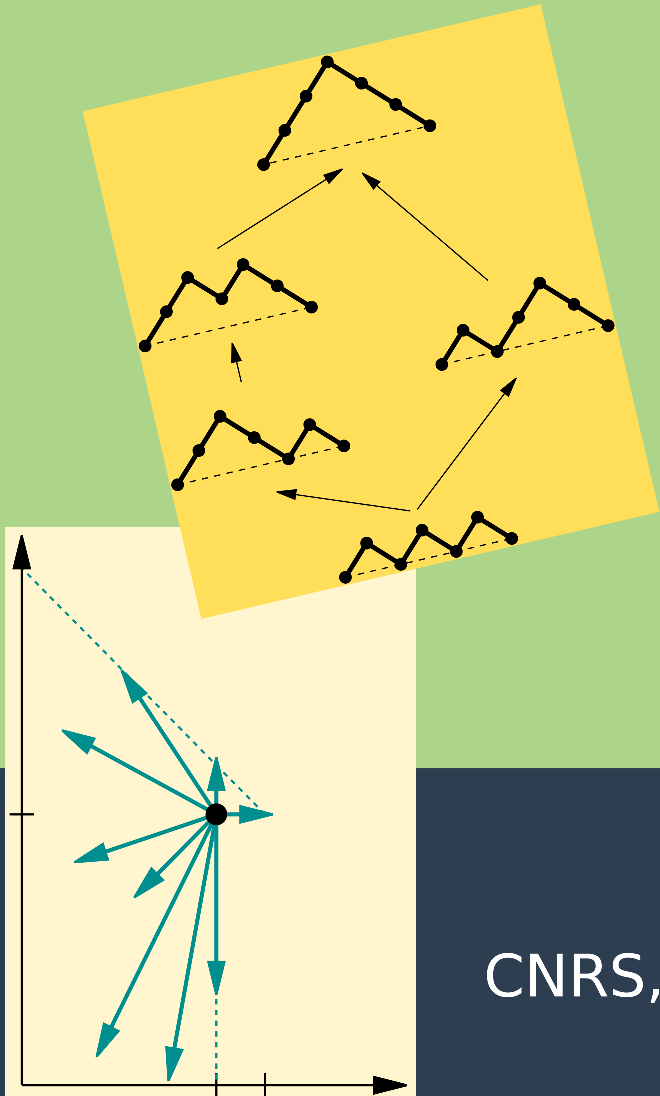
Quadrant walks and the ascent poset on Dyck paths



with Jean-Luc Baril, Sergey Kirgizov (LIB, Dijon) and Mehdi Naima (LIP6, Paris)

Mireille Bousquet-Mélou
CNRS, LaBRI, Université de Bordeaux

arXiv:2409.15982

A press review

Indeed, walks in the quarter plane naturally encode many combinatorial objects (certain trees, maps, permutations, Young tableaux, etc.).
[Fayolle-Raschel 15]

This question is ubiquitous since lattice walks encode several classes of mathematical objects in discrete mathematics (permutations, trees, planar maps...), in statistical physics (magnetism, polymers...), in probability theory (branching processes, games of chance...), in operations research (birth-death processes, queueing theory).
[Bonnet-Hardouin 24(a)]

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The enumeration of quadrant walks with small steps appears as an essential step in the study of discrete partial differential equations, because the resulting equations are as simple as possible: linear, first order, and involving derivatives with respect to two variables only.
[mbm 16]

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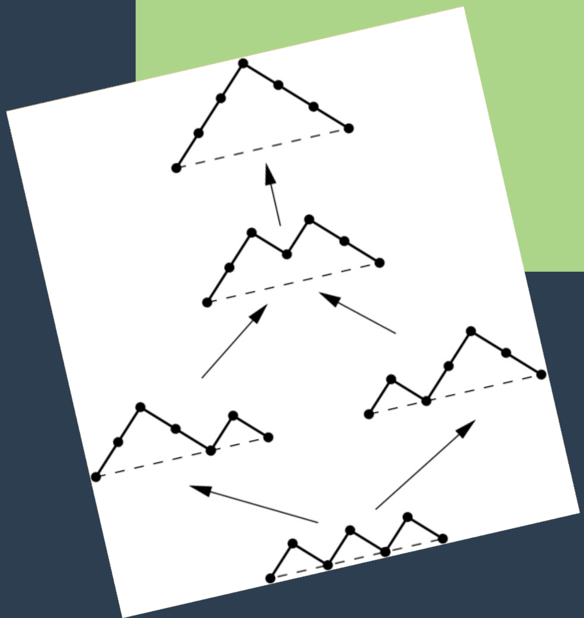
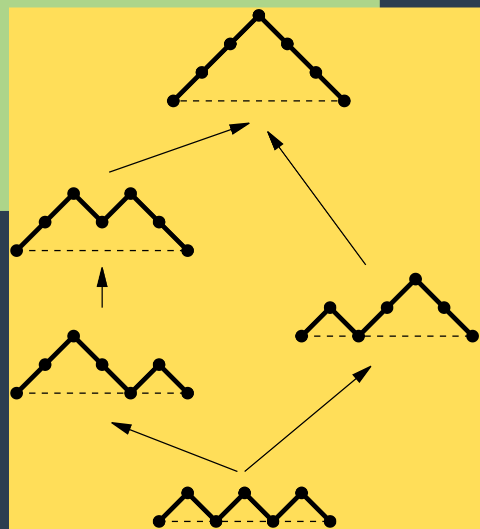
An illustration

The study of the associated equations may provide tools to study similar equations

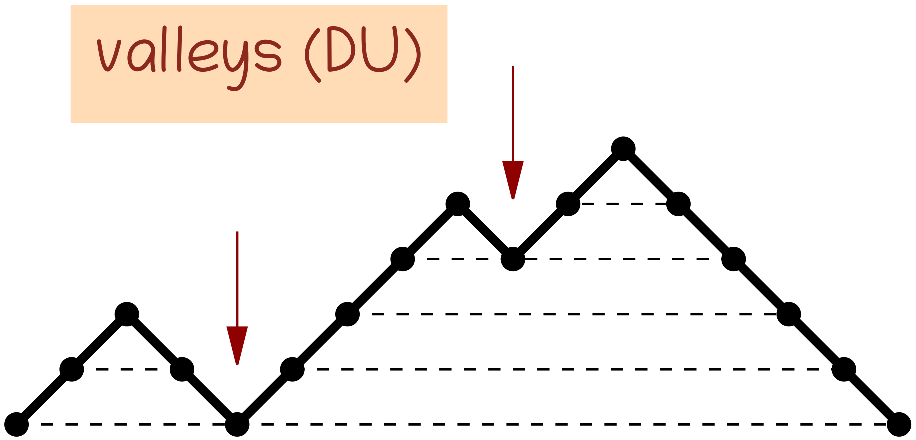
I. Two orders on Dyck paths





Dyck paths = discrete excursions

- A Dyck path of size n=8 (size=number of up steps)



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The simplest poset: Stanley's lattice

- A poset on Dyck paths with n up steps

Poset = partially ordered set

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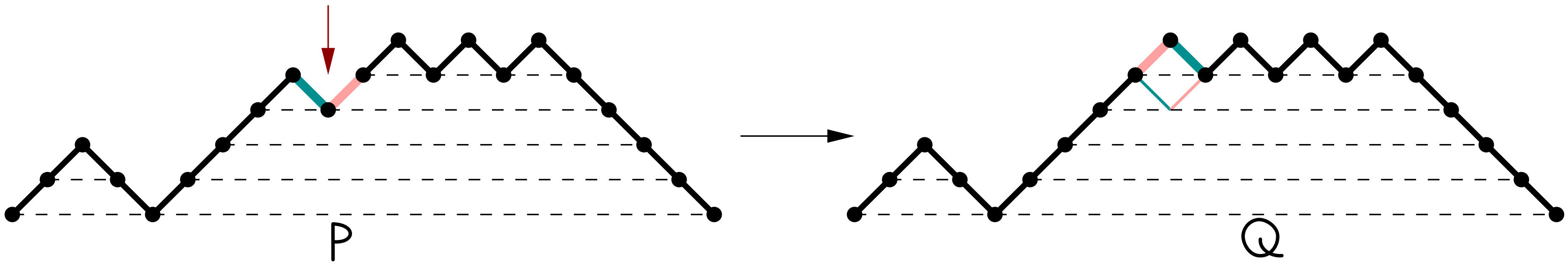

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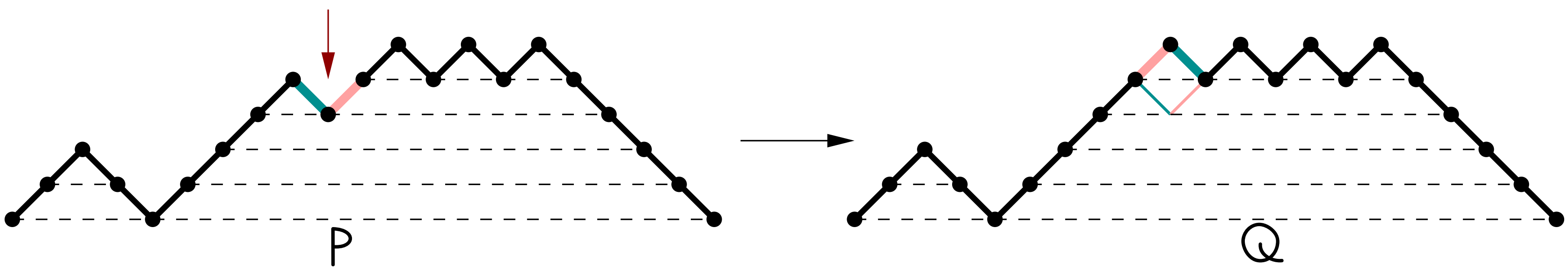

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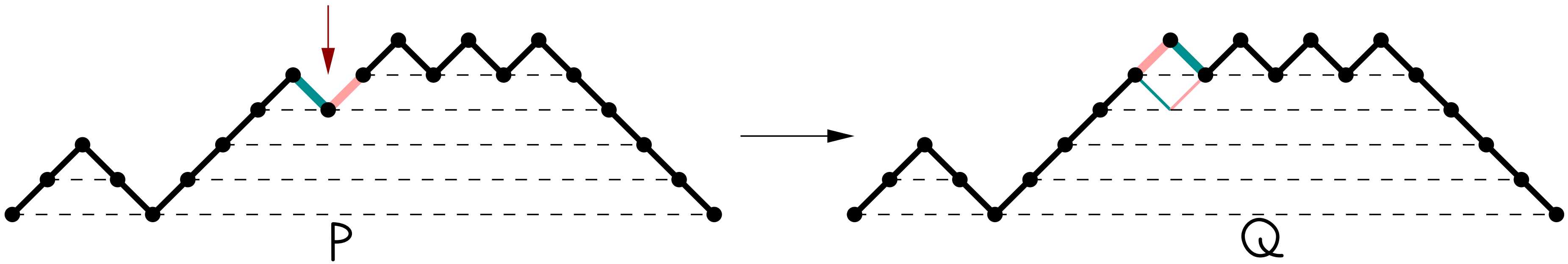

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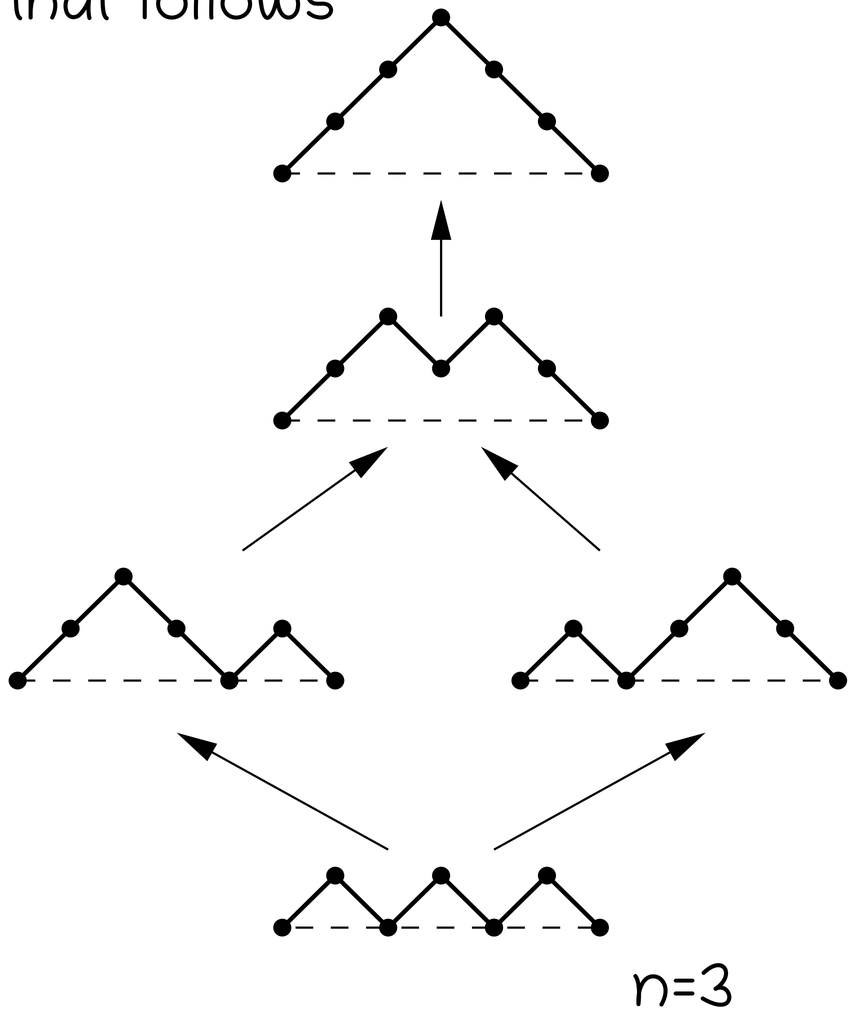

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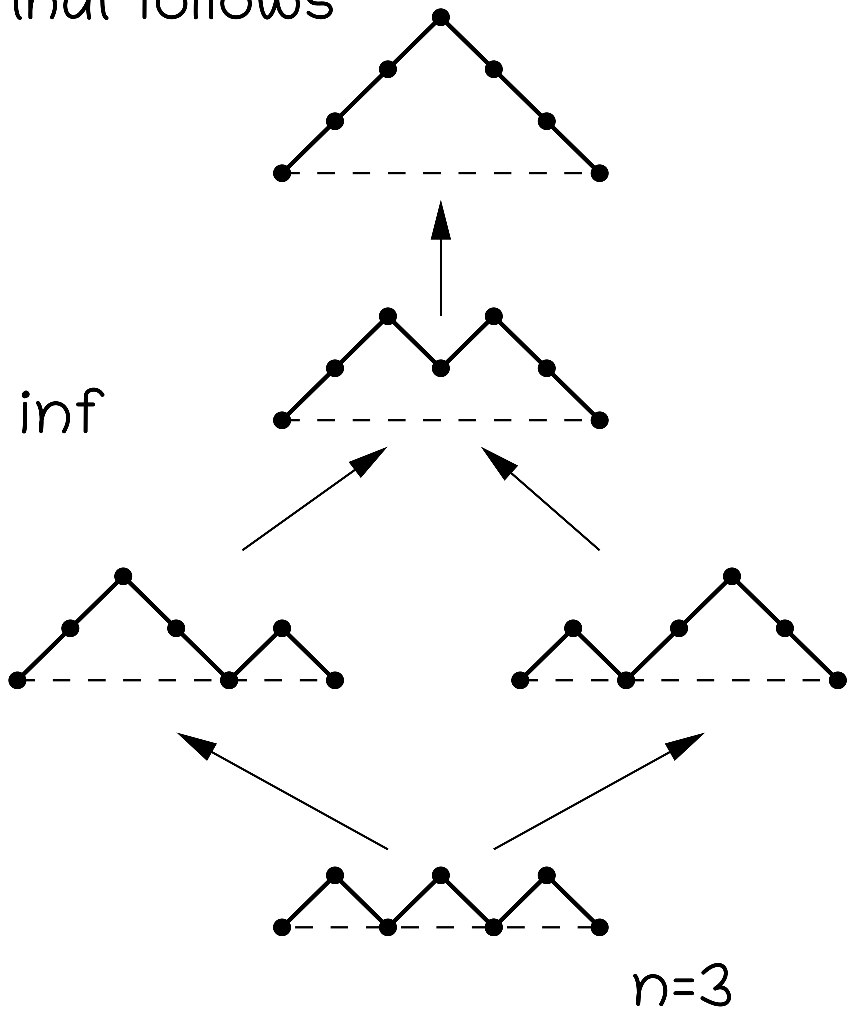
Characterization: $P \leq Q$ if P lies below Q

- Lattice structure: existence of sup and inf

Poset = partially ordered set



The ascent poset (or: greedy Stanley lattice?)

- A poset on Dyck paths with n up steps
- **Cover relations:** choose a valley in the path P.

  Swap the down step and the ascent that follows

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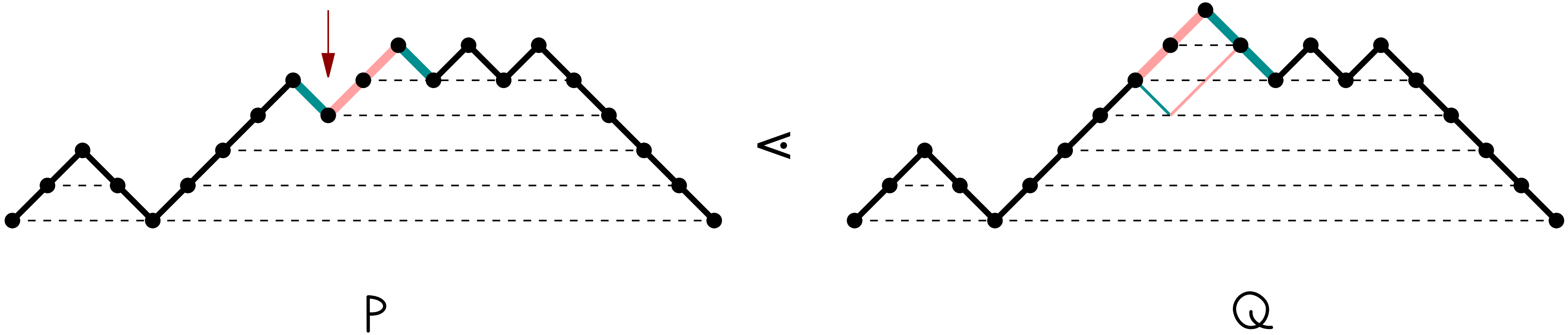
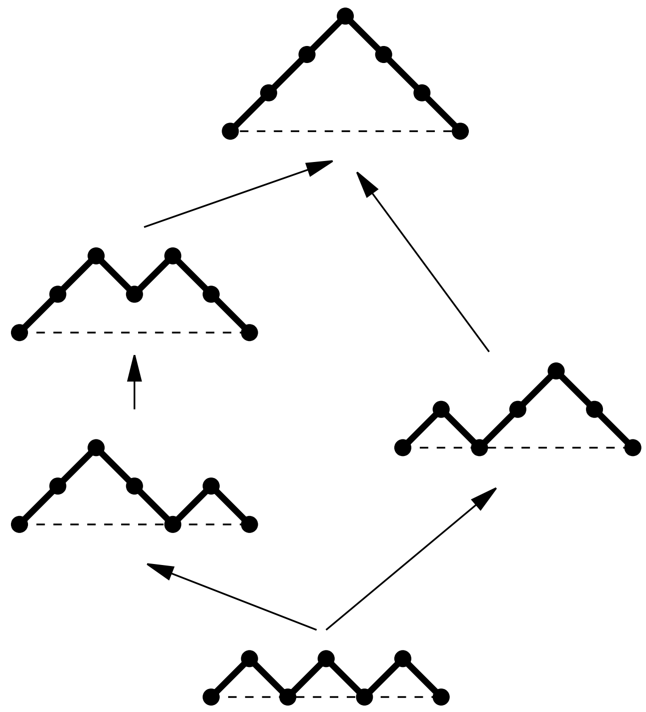
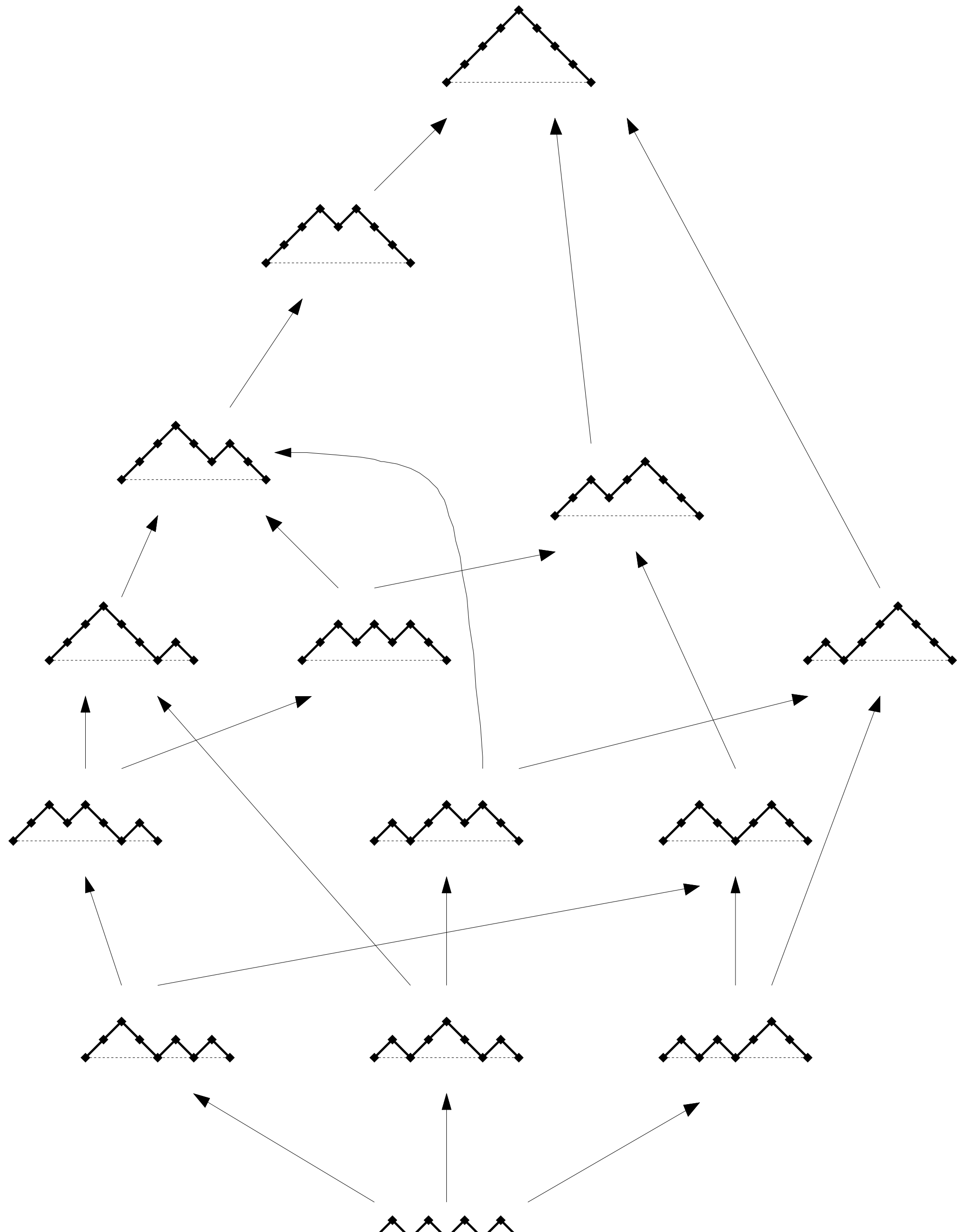


[Chenevière, Nadeau...]

Ascent posets: n = 3, 4





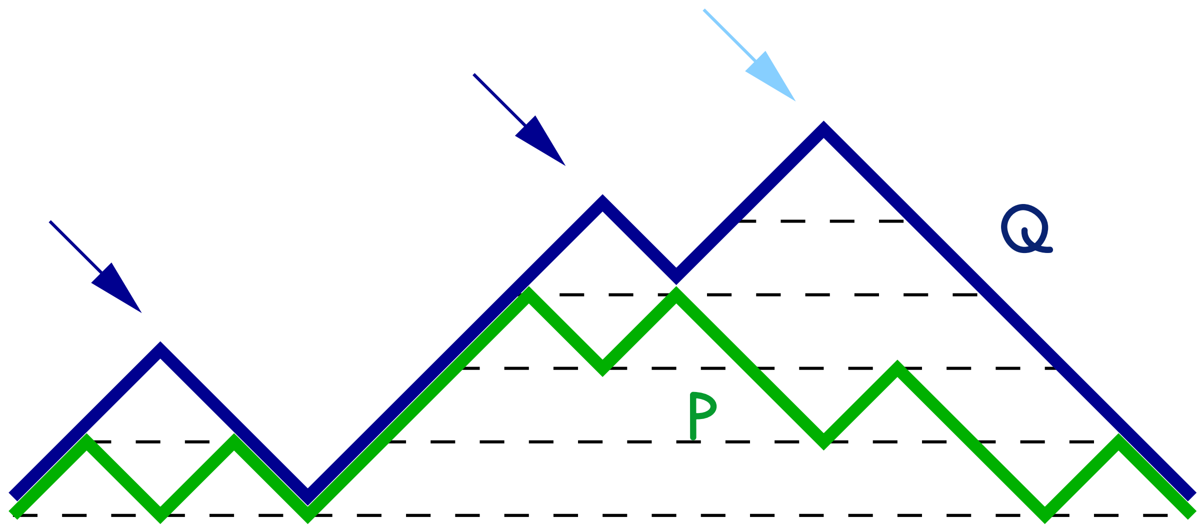
II. The number of intervals

Interval $[P,Q] \sim (P,Q)$ with $P \preccurlyeq Q$

A characterization of the ascent order

Proposition. In the ascent poset, $P \preccurlyeq Q$ iff

- P lies below Q
- every descent of Q **is** (i.e. lies on the same diagonal as) a descent of P.



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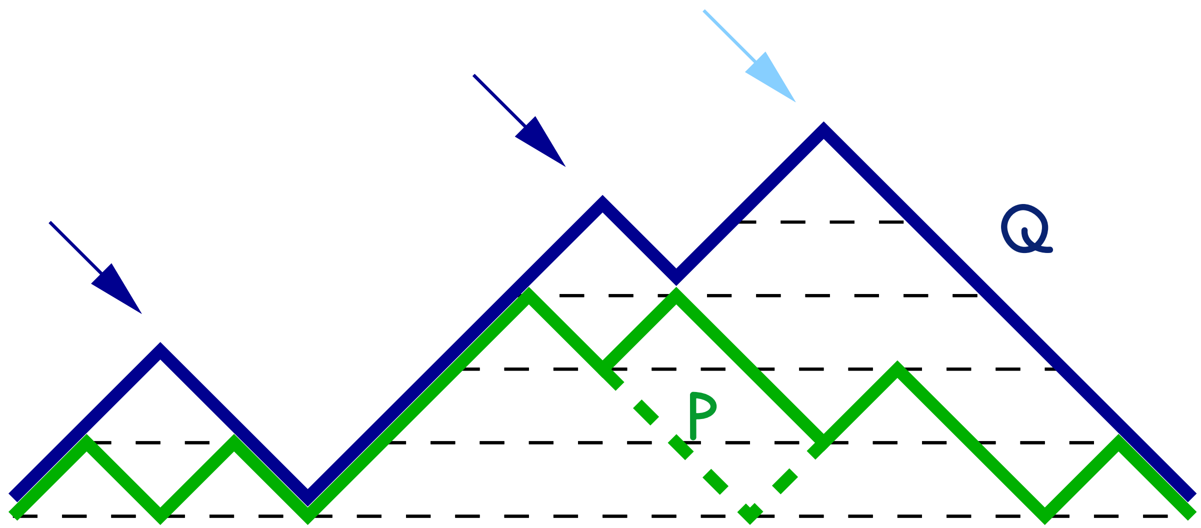

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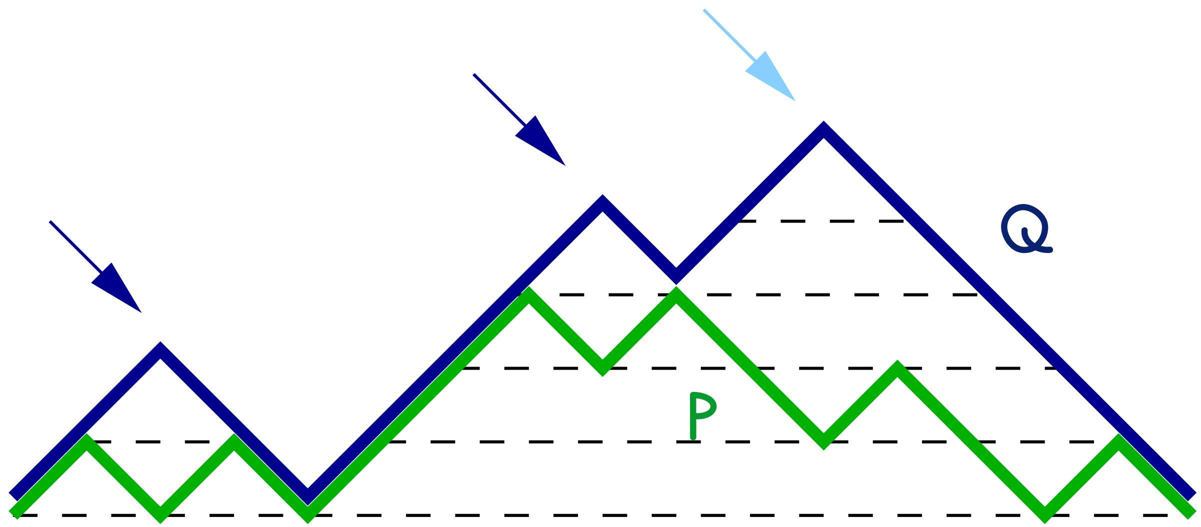

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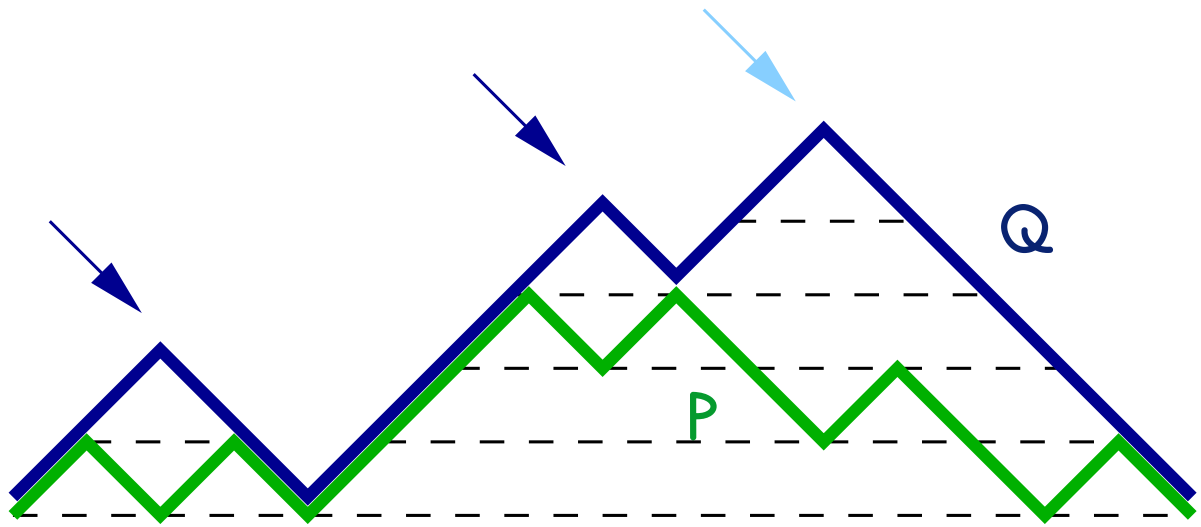

Corollary: if [P,Q] is an interval, deleting the last peak of P and the last peak of Q gives a new interval.

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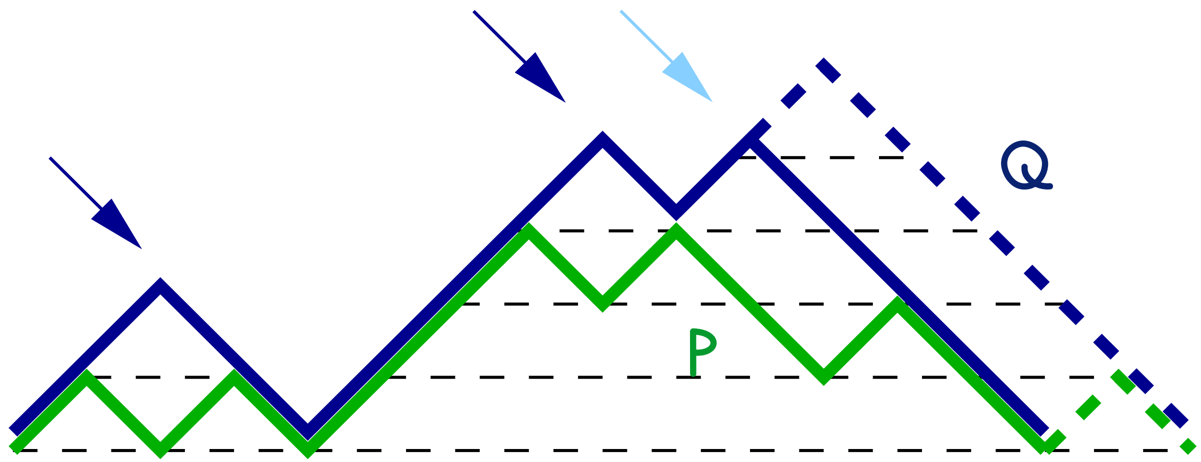

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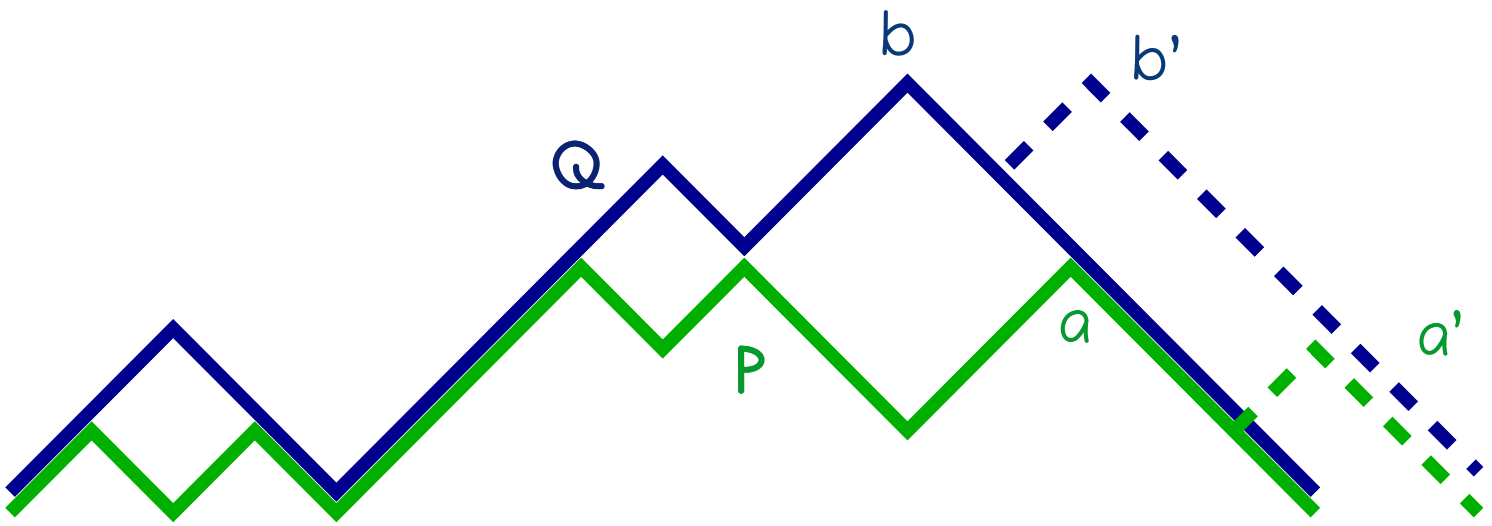
Conversely, starting from an interval [P,Q] with final peaks at heights $a \leq b$, adding peaks in P and Q at heights a' and b' gives an interval iff...



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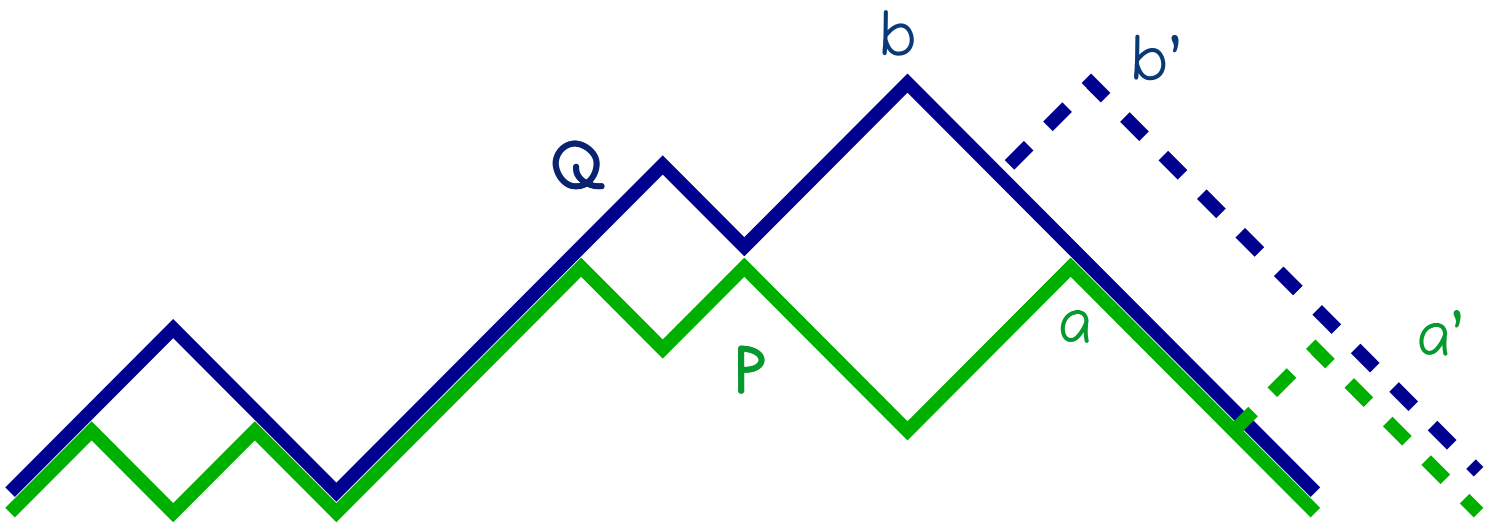

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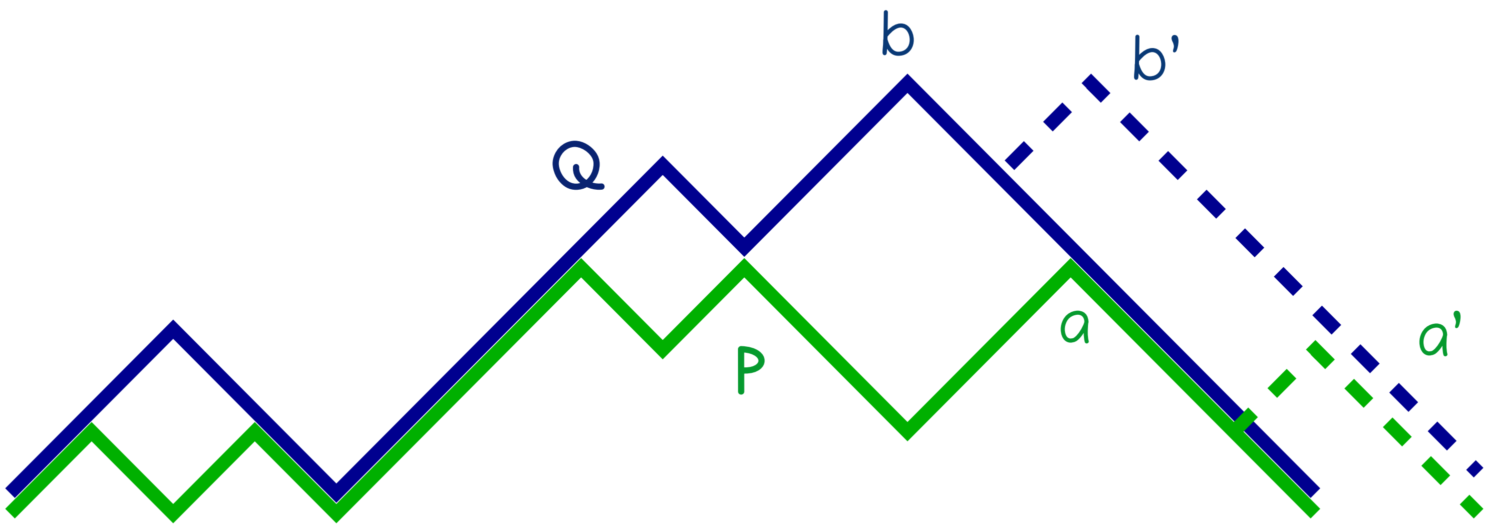

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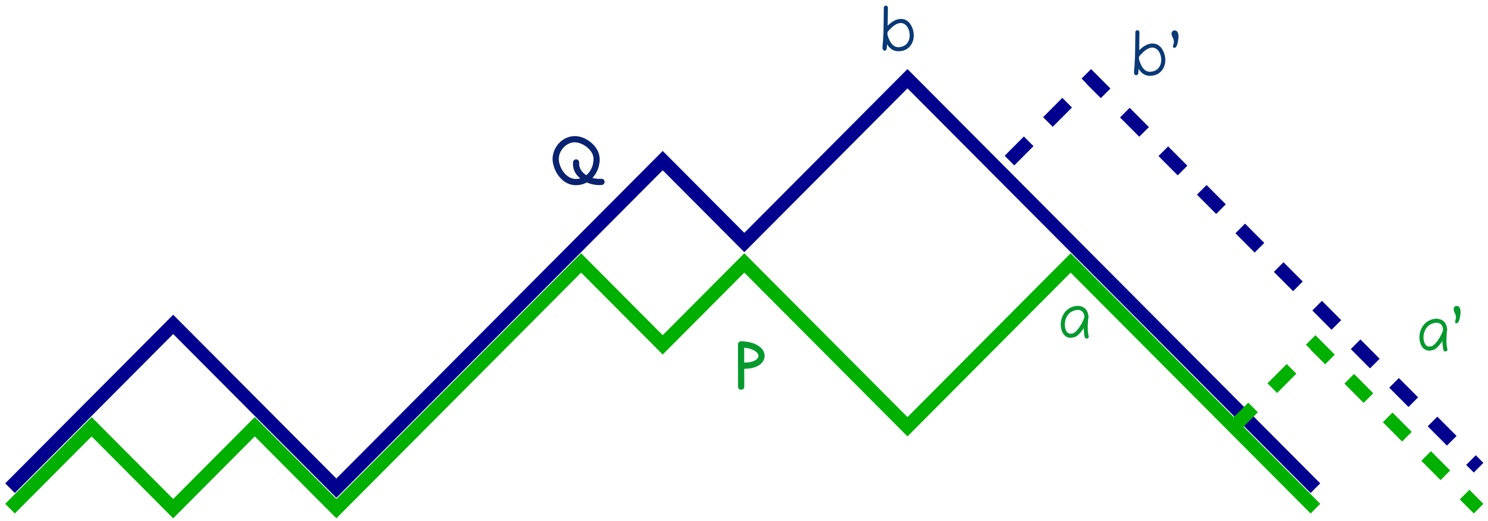

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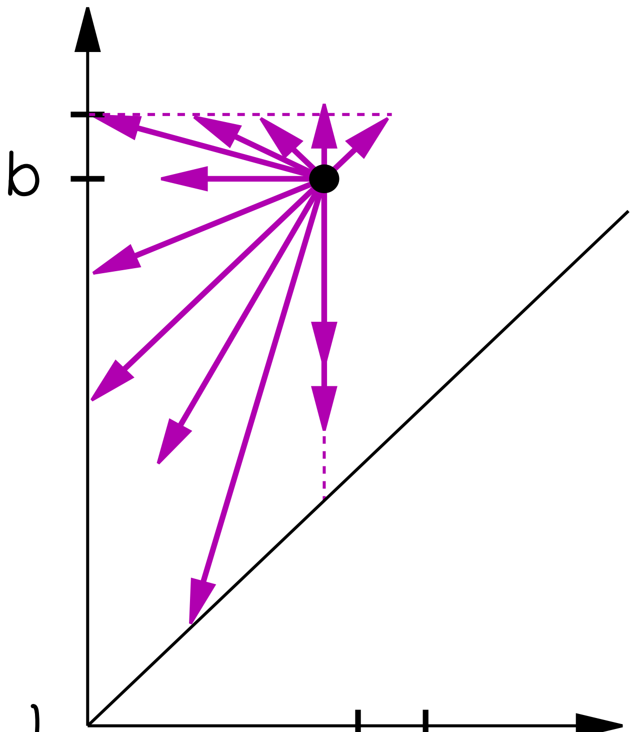

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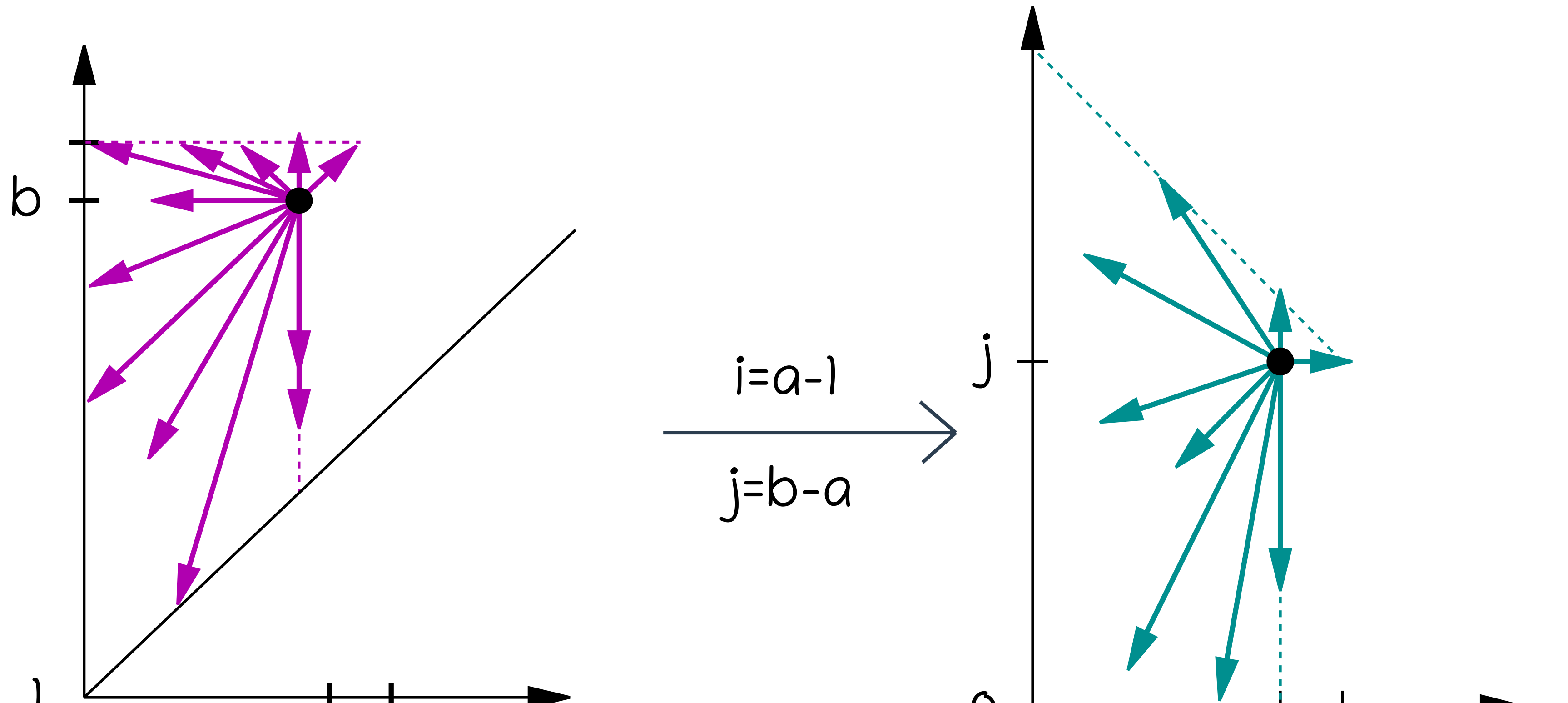

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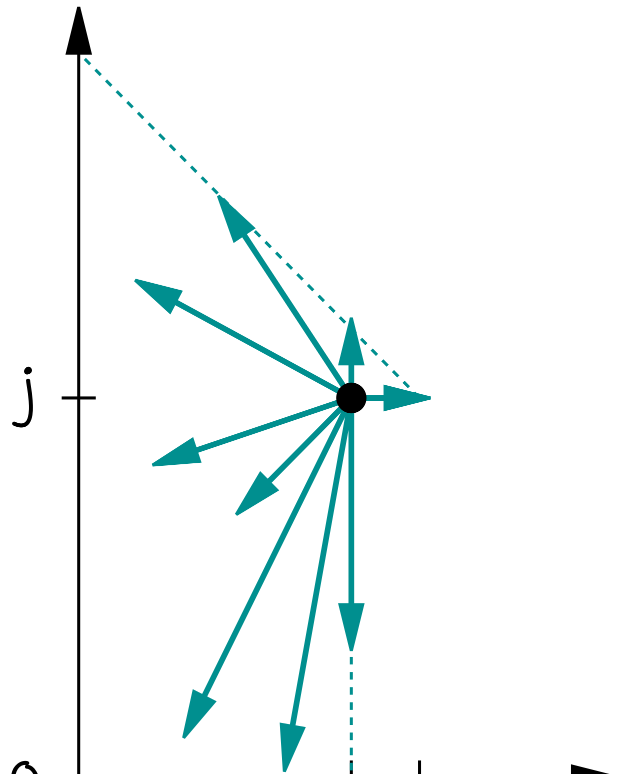
Bijection

intervals of size n

$\approx$

quadrant walks of length n-1 starting from (0,0)



Recursive construction of ascent intervals

- Let Q(t;x,y)=Q(x,y) be the GF of the associated quadrant walks:

$$Q(x,y) = \sum_{w} t^{|w|} x^{i(w)} y^{j(w)}.$$

Then the GF of ascent intervals is G=tQ(1,1).

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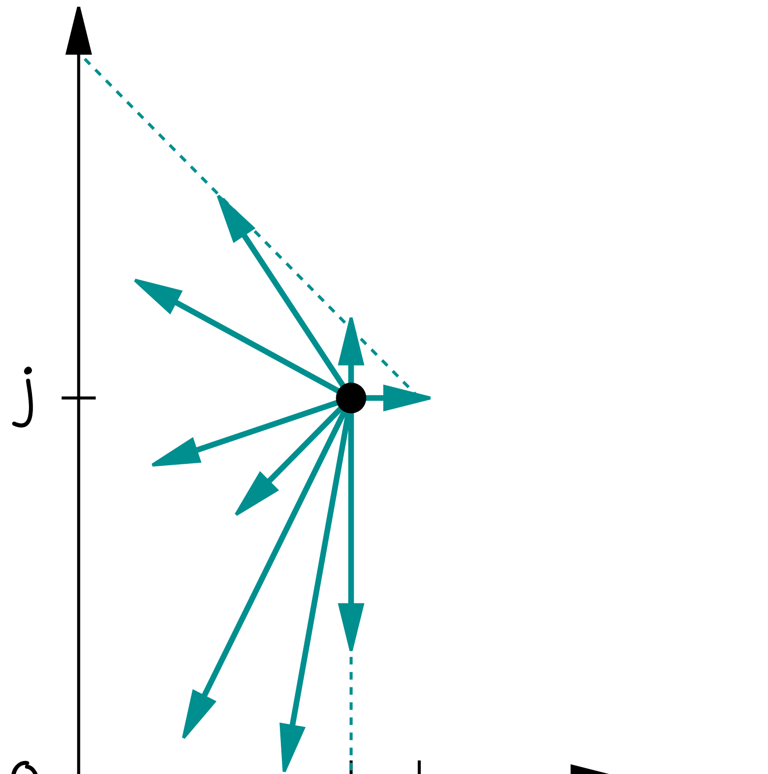

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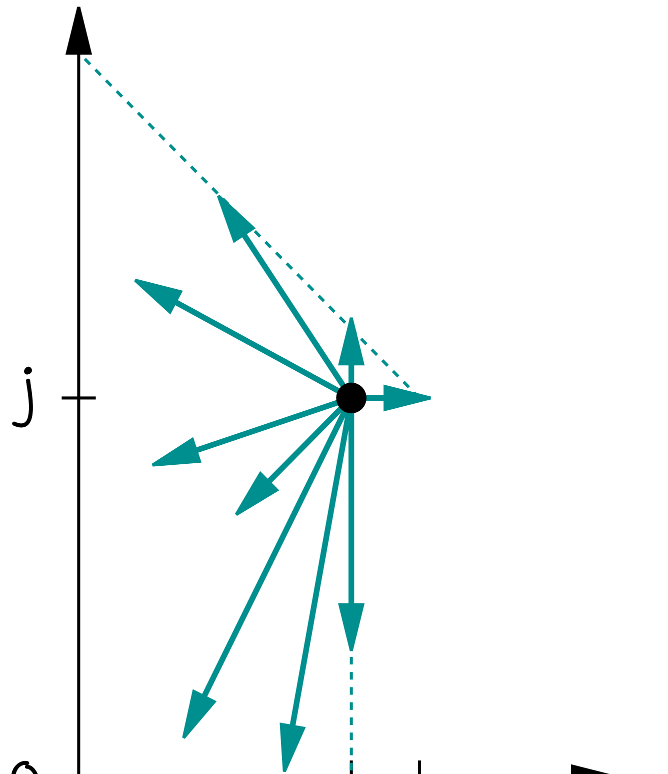

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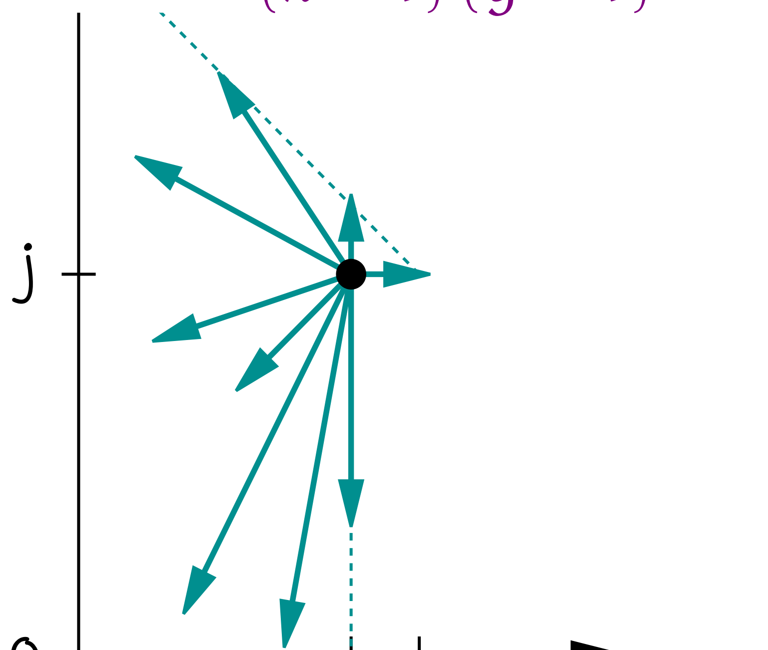
$$Q(x,y) = 1 + txQ(x,y) + ty^2 \frac{xQ(x,y) - yQ(y,y)}{(x-y)(y-1)} - t\frac{xQ(x,1) - Q(1,1)}{(x-1)(y-1)}.$$



A functional equation with two catalytic variables

- The GF of ascent intervals is $tQ(1,1)$, where $Q(x,y)=Q(t;x,y)$ satisfies:

$$K(x,y)(y-1)Q(x,y) = y - 1 - \frac{ty^3}{x-y}Q(y,y) - t\,\frac{xQ(x,1)-Q(1,1)}{x-1}$$

with kernel

$$K(x,y) = 1 - tx - \frac{txy^2}{(x-y)(y-1)}.$$

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- **Well understood** (algebraic/differential properties): quadrant walks **with finitely many small steps**

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Very few algebraic cases
(solution of polynomial equation)
Only 4 with small steps

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Thm. Ascent intervals have **an algebraic GF**, namely

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Asymptotics:

$$g(n) \sim \kappa\,\mu^n n^{-7/2}, \qquad \text{with} \qquad \mu = \frac{11+5\sqrt{5}}{2}.$$

III. Algebraicity via Tutte's invariants

d'après [Bernardi, mbm, Raschel 21]

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- **Observation**: an equation of the form

$$K(x,y)H(x,y) = I(x) - J(y)$$

would probably be easier to solve.
The pair (I(x),J(y)) is **a pair of invariants**.

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Remark: there should *really* be a factor $K(x,y)$ on the LHS.

Proving algebraicity via invariants

Strategy [Bernardi, mbm, Raschel 21]

(1) Construct **rational invariants** $(I_0(x), J_0(y))$ **from the kernel** (finite group)

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(3) **There are few invariants:** $I_0(x)$ and $I_1(x)$ must be polynomially related over $\mathbb{Q}((t))(x)$ $\rightarrow$ an equation for $Q(x,1)$ (and specializations) with no y

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(4)The solutions of such equations are **algebraic** [mbm-Jehanne 06]

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Strategy [Bernardi, mbm, Raschel 21]

(1) Construct **rational invariants** $(I_0(x), J_0(y))$ **from the kernel** (finite group)

(2) Construct another pair of invariants $(I_1(x), J_1(y))$ **involving the series $Q(x,1)$ and $Q(y,y)$** from the equation

(3) **There are few invariants:** $I_0(x)$ and $I_1(x)$ must be polynomially related over $\mathbb{Q}((t))(x)$ → an equation for $Q(x,1)$ (and specializations) with no y

(4) The solutions of such equations are **algebraic** [mbm-Jehanne 06]

$$K(x,y)H(x,y) = I(x) - J(y)$$

cf. [Bonnet, Hardouin 24(a)]

(I) Constructing invariants from the kernel

The kernel:

$$K(x, y) = 1 - tx - \frac{txy^2}{(x-y)(y-1)}.$$

Let

$$I_0(x) = \frac{1}{1-tx} - \frac{1}{tx^2} + \frac{1+t}{tx} + x(1-t) - tx^2$$

$$J_0(y) = -\frac{t}{(y-1)^2} + \frac{1-t}{y-1} - \frac{1}{ty^2} + \frac{1+t}{yt} + y.$$

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This is **a pair of invariants**:

$$I_0(x) - J_0(y) = \frac{(x-y)(1-y+txy)(x+y-xy-xyt(1+x-xy))}{x^2y^2t(xt-1)(y-1)} K(x,y).$$

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Construction?

A **group of order 10** generated by two birational involutions of (x,y) leaves the kernel unchanged.

Play with the group and the roots of the kernel.

(2) Constructing invariants from the equation

$$K(x,y)(y-1)Q(x,y) = y - 1 - \frac{ty^3}{x-y}Q(y,y) - t\,\frac{xQ(x,1)-Q(1,1)}{x-1}$$

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Construction: decoupling (multiplicative then additive)

(3) Relating invariants

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(no pole at y=0, 1) is **independent of y**, and thus equal to

$$2 - 4t - 2t^2 Q(1,1)$$

(value at y=1).

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(4) An equation for Q(y,y) -- Algebraicity

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$$\hookrightarrow \; y^2 t^2 (y-1)^2 Q(y,y)^2 + \left(y\left(2y^2-5y+1\right)t - (y-1)(y-2)\right) Q(y,y)$$
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- Systematic algebraic solution [Brown 65, mbm-Jehanne 06]

$$64\, t^6 Q_{11}^3 + 16t^3\left(11t^2 - 18t - 1\right) Q_{11}^2 + \left(161t^4 - 452t^3 + 238t^2 - 28t + 1\right) Q_{11} \\ + 49\, t^3 - 167t^2 + 25t = 1.$$

IV. More posets, more walks

m-Dyck paths and mirrored m-Dyck paths

Two sub-posets of the ascent poset of size mn, and their intervals

➔ **m-Dyck paths**: last peak decomposition

$$Q(x,y) = 1 + tx^m Q(x,y) + ty^2 \frac{x^m Q(x,y) - y^m Q(y,y)}{(x-y)(y-1)} - t\frac{x^m Q(x,1) - Q(1,1)}{(x-1)(y-1)}$$

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→ **Explicit asymptotic results** ⇒ not algebraic, not D-finite for m>1.
[Denisov, Wachtel 15]

Final remarks

- **Combinatorial proof** for the number/GF of ascent intervals? (m=1)

$$(n+4)(2n+7)\,g(n+2) = 2\left(11n^2+44n+42\right)g(n+1) + n\,(2n+1)\,g(n)$$

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Thanks for your attention