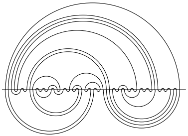
Meandric systems and tree-indexed Catalan sums

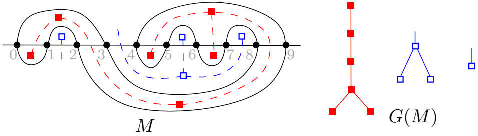
joint work with A. Bostan and V. Féray

Paul Thévenin

Un quart de siècle pour un quart de plan

April 16, 2025





Tree-indexed Catalan sums

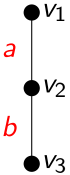
Given a tree $T := (V, E)$, define the sum

$$S_T = \sum_{(x_e) \in \mathbb{Z}_+^E} \prod_{v \in V} C_{X_v} 4^{-X_v}$$

where:

- $X_v = \sum_{e \ni v} x_e$,
- $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number.

Examples



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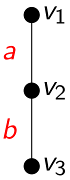	$X_{v_1} = a, X_{v_2} = a + b, X_{v_3} = b$

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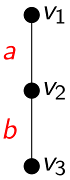	$X_{v_1} = a, X_{v_2} = a + b, X_{v_3} = b$ $S_T = \sum_{a,b\geq 0} C_a C_{a+b} C_b 4^{-2a-2b}$

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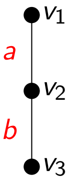
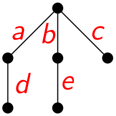
	$X_{v_1} = a, X_{v_2} = a + b, X_{v_3} = b$ $S_T = \sum_{a,b \geq 0} C_a C_{a+b} C_b 4^{-2a-2b}$
	

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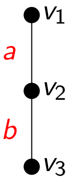
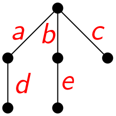
	$X_{v_1} = a, X_{v_2} = a + b, X_{v_3} = b$ $S_T = \sum_{a,b \geq 0} C_a C_{a+b} C_b 4^{-2a-2b}$
	$S_T = \sum_{\mathbb{Z}_+^7} C_{a+b+c} C_{a+d}$ $C_{b+e} C_c C_d C_e 4^{-\cdots}$

Main result

Theorem [Bostan, Féray & T. '25+]

For any tree T, we have

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For any tree T, we have

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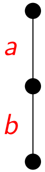
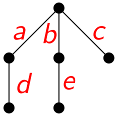
- We also provide a constructive algorithm to compute these sums.
- Mathematica, Maple do not manage to compute them!

Examples





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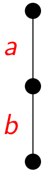
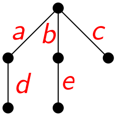
	$S_T = \sum_{a,b \geq 0} C_a C_{a+b} C_b 4^{-2a-2b}$ $= \boxed{8 - \frac{64}{3\pi}}$
	

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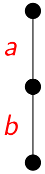

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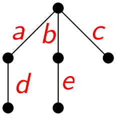


$$S_T = \sum_{\mathbb{Z}_+^7} C_{a+b+c} C_{a+e} C_{b+f} C_c C_d C_e 4^{-\cdots}$$

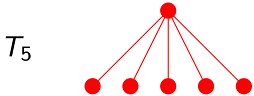
$$= \boxed{-208 + \frac{7552}{9\pi} - \frac{2304}{\pi^2} + \frac{16384}{3\pi^3}}$$

Proof: induction on the size of the tree

- Base case: trees of size 1;
- Induction step: generalized *decorated trees*.

Base case: trees of size 1

For $k \in \mathbb{Z}_+$, let T_k be the star tree with k branches.



$$S_{T_k} = \sum_{x_1,\dots,x_k \geq 0} C_{x_1} \cdots C_{x_k} C_{x_1+\dots+x_k} 4^{-2x_1-\dots-2x_k}.$$

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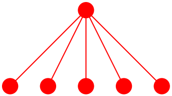
T_5



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Theorem [Bostan, Féray & T. '25+]

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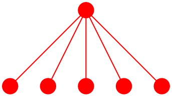

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Theorem [Bostan, Féray & T. '25+]

$S_{T_1} = \frac{16}{\pi} - 4$, $S_{T_2} = 8 - \frac{64}{3\pi}$ and, for all $k \geq 3$:

$$S_{T_k} = \frac{64}{\pi} \cdot \left(\sum_{\ell=0}^{k-3} \binom{k-3}{\ell} \frac{1}{(2\ell+1)(2\ell+3)(2\ell+5)} \right).$$

Ideas of proof: hypergeometric functions

- Sums S_T for trees of height 1 can be directly expressed in terms of hypergeometric functions.

Standard hypergeometric function

The standard hypergeometric function ${}_2F_1(a, b; c; z)$ is defined as:

$$ {}_2F_1(a, b; c; z) := \sum_{n \geq 0} \frac{a^{\uparrow n} b^{\uparrow n}}{c^{\uparrow n}} \frac{z^n}{n!}, $$

where $x^{\uparrow n} := x(x+1) \cdots (x+n-1)$ is the n-th raising power of x.

An example: the star tree T_1

$$S_T = \sum_{a \geq 0} C_a^2 4^{-2a}$$

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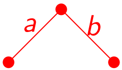
$$\begin{aligned} S_T &= \sum_{a\geq 0} C_a^2 4^{-2a} \\ &= 4\ {}_2F_1(-\frac{1}{2}, -\frac{1}{2}; 1; 1) - 4. \\ &= \frac{16}{\pi} - 4 \end{aligned}$$

[Gauss, a long time ago.]

Another example and the Catalan identity



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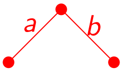

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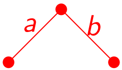

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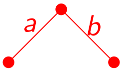

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$$= \sum_{x\geq 0} C_x C_{x+1} 4^{-2x} (\text{Catalan identity}).$$

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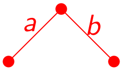

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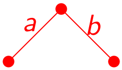

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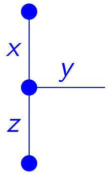
Trees with one half-edge

- The result also holds for T a tree with an additional half-edge.



$$S_T = \sum_{x,y,z\geq 0} C_x C_{x+y+z} C_z 4^{-2x-y-2z} = \frac{32}{3\pi}.$$

Formal power series version

Our result actually holds at the level of formal power series. For a tree T, define

$$S_T(t) := \sum_{(x_e)\in\mathbb{Z}_+^E} \prod_{v\in V} C_{X_v} t^{X_v}.$$

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Theorem [Bostan, Féray & T. '25+]

For any tree T, we have

$$S_T(t) \in \mathbb{Q}\left[t^{\pm 1}, \sqrt{1-4t}, {}_2F_1(-\frac{1}{2}, -\frac{1}{2}; 1; 16t^2), {}_2F_1(-\frac{1}{2}, \frac{1}{2}; 2; 16t^2)\right].$$

Gauss' identity

Lemma [Gauss]

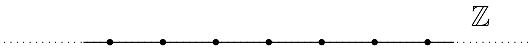
If $c - a - b > 0$, we have

$$ {}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. $$

- Consequently, ${}_2F_1(-\frac{1}{2}, -\frac{1}{2}; 1; 1) = \frac{4}{\pi}$, ${}_2F_1(-\frac{1}{2}, \frac{1}{2}; 2; 1) = \frac{8}{3\pi}$.

Motivation: the infinite noodle

- Draw i.i.d. arrows on $\mathbb{Z}$, pointing to the left/right with proba $1/2$.
- Connect them in the unique noncrossing way.



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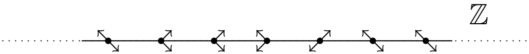

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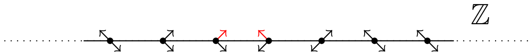

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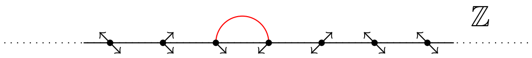

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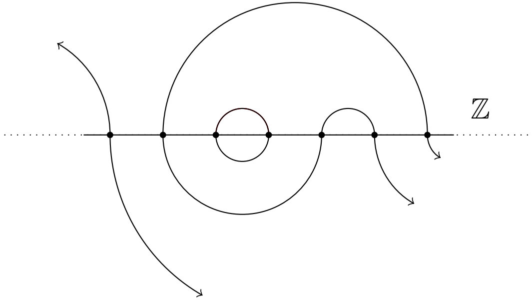

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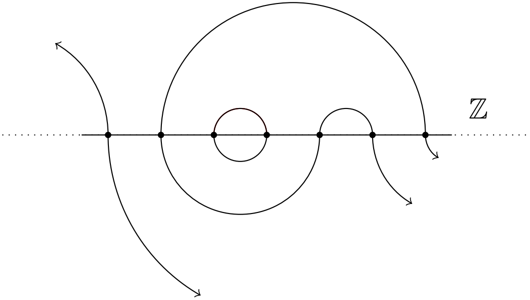

Is there an infinite component?

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Theorem [Curien, Kozma, Sidoravicius & Tournier '19]

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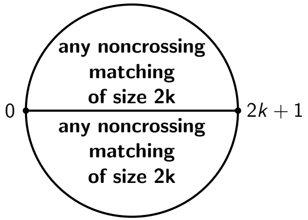
Conjecture [Borga, Gwynne, Park '23]

$$\mathbb{P}(|L(0)| \geq k) \underset{k\to\infty}{\sim} k^{-\frac{2\sqrt{2}-1}{7}+o(1)}.$$

Possible shapes and dual trees (1)

- Consider $\mathbb{P}(|L(0)| = 2)$.



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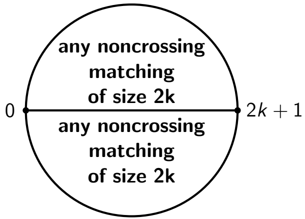

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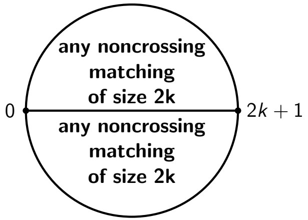

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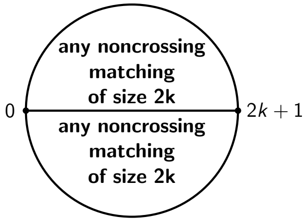

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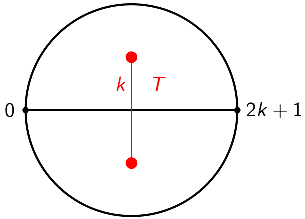

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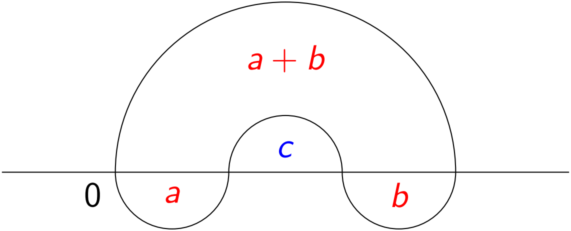
$$\mathbb{P}(|L(0)| = 2) = \frac{1}{16} \sum_{k \geq 0} C_k^2 4^{-2k} = \frac{1}{16} S_T$$

Possible shapes and dual trees (2)



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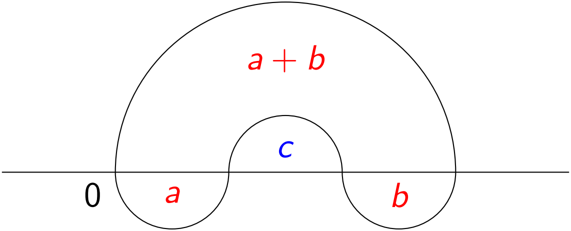
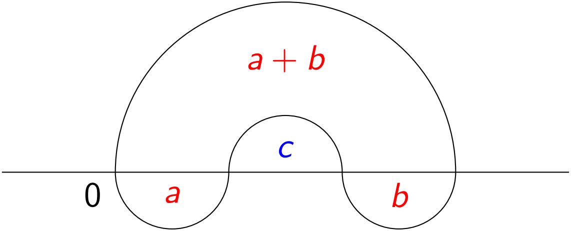


We "fill in" the component with:

- noncrossing matchings of sizes $2a, 2b, 2(a+b)$;
- a noncrossing matching of size $2c$.

Possible shapes and dual trees (2)



We "fill in" the component with:

- noncrossing matchings of sizes $2a, 2b, 2(a+b)$;
- a noncrossing matching of size $2c$.

Any such configuration occurs with probability $\left(\frac{1}{4}\right)^{2a+2b+c+4}$.

$$\mathbb{P}(L(0) = S) = \frac{1}{4^4} \sum_{a,b,c \geq 0} C_a C_b C_{a+b} C_c 4^{-2a-2b-c}.$$

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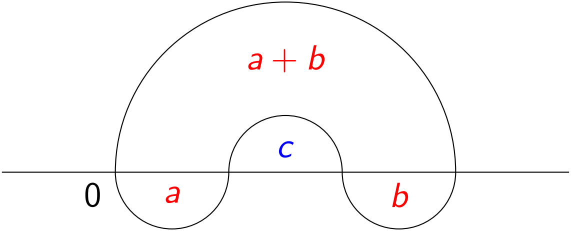

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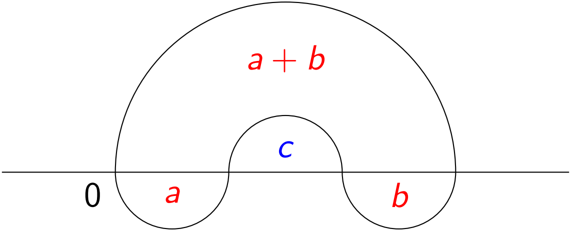

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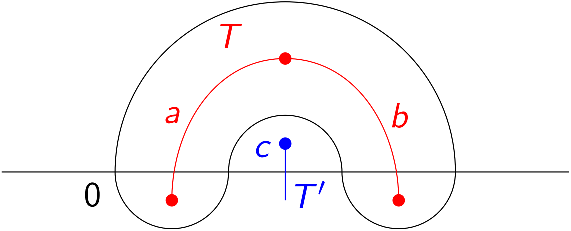

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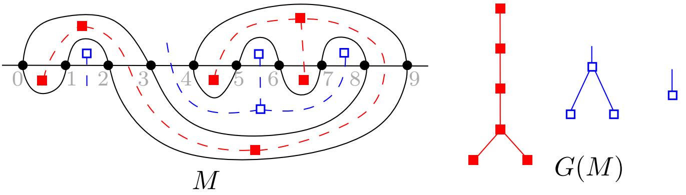
$$= \frac{1}{4^4} S_T S_{T'}$$

General shapes

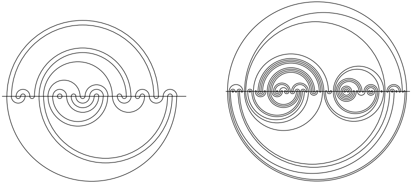
In order to compute $\mathbb{P}(|L(0)| = k)$:

- Sum over all possible shapes of size k ;
- For a given shape S of size k, $\mathbb{P}(L(0)$ has shape $S)$ can be expressed as the product of the sums associated to the *dual trees* of the shape S.



Motivation (2): meandric systems and walks in the quarter plane

- A meandric system of size n is a pair of noncrossing matchings on $2n$ points.



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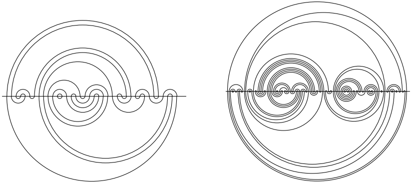


Theorem [Féray, T. '22]

Let M_n be a random uniform meandric system of size n. Then, M_n converges locally towards the infinite noodle.

Motivation (2): meandric systems and walks in the quarter plane

As a consequence, we get

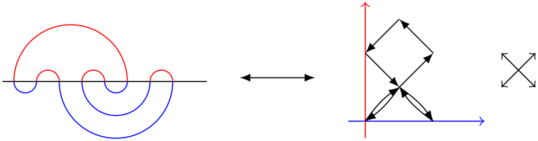
Corollary [Féray, T. '22]

Let M_n be a random uniform meandric system of size n. For any fixed $k \geq 0$, let $N_k(M_n)$ be the number of loops of M_n containing k points. Then:

$$\frac{N_k(M_n)}{2n} \underset{n\to\infty}{\overset{(\mathbb{P})}{\to}} \frac{1}{k}\mathbb{P}(|L(0)| = k).$$

Motivation (2): meandric systems and walks in the quarter plane

- We can code a meandric system by an excursion in $\mathbb{N}^2$ with diagonal small steps.



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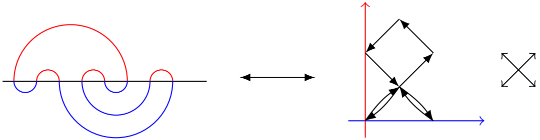
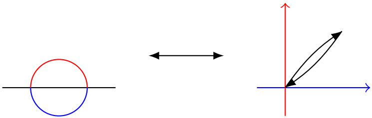
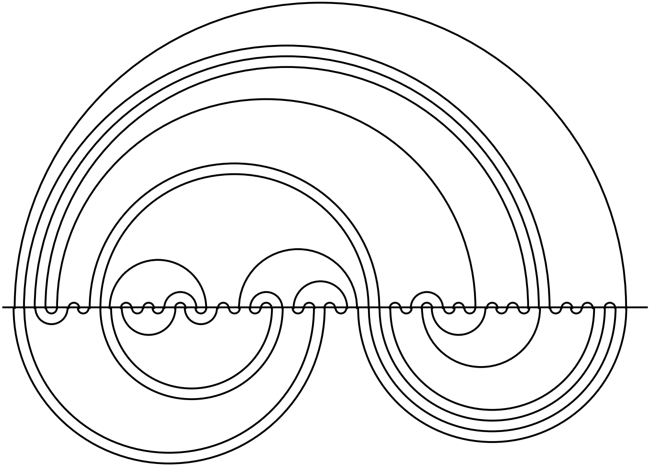


- The numbers $\mathbb{P}(|L(0)| = k)$ count (asymptotically) patterns in excursions.



Meanders [Poincaré, 1912]

- Meander = meandric system with only one connected component.



Meanders [Poincaré, 1912]

Conjecture [Di Francesco, Golinelli, Guitter '00]

Let $\mathcal{M}_n$ be the number of meanders of size n. Then, as $n \to \infty$:

$$\mathcal{M}_n \sim C\, R^{2n} n^{-\alpha},$$

for some constants $C, R > 0$ and

$$\alpha = \frac{29 + \sqrt{145}}{12}.$$

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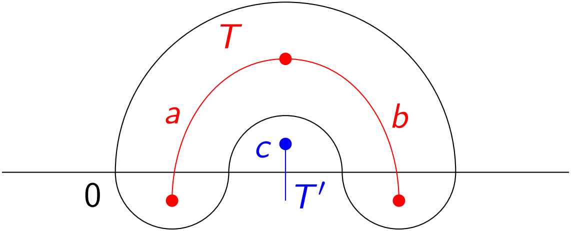
- Counting walks in the quarter plane avoiding infinitely many patterns.

THANKS



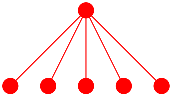
Proof: an induction on the size of the tree

- We want to prove that, for any tree T, $S_T \in \mathbb{Q}[\frac{1}{\pi}]$.
- Base case: trees of size 1;
- Induction step: generalized *decorated trees*.

Base case: trees of size 1

For $k \in \mathbb{Z}_+$, let T_k be the star tree with k branches.

T_5



$$S_{T_k} = \sum_{x_1,\ldots,x_k \geq 0} C_{x_1} \cdots C_{x_k} C_{x_1+\ldots+x_k} 4^{-2x_1-\cdots-2x_k}.$$

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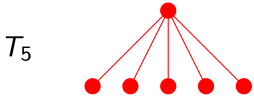

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Theorem [Bostan, Féray & T. '25+]

$S_{T_1} = \frac{16}{\pi} - 4$, $S_{T_2} = 8 - \frac{64}{3\pi}$

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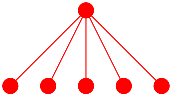

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Theorem [Bostan, Féray & T. '25+]

$S_{T_1} = \frac{16}{\pi} - 4$, $S_{T_2} = 8 - \frac{64}{3\pi}$ and, for all $k \geq 3$:

$$S_{T_k} = \frac{64}{\pi} \cdot \left(\sum_{\ell=0}^{k-3} \binom{k-3}{\ell} \frac{1}{(2\ell+1)(2\ell+3)(2\ell+5)} \right).$$

Ideas of proof: hypergeometric functions

- Sums S_T for trees of height 1 can be directly expressed in terms of hypergeometric functions.

Standard hypergeometric function

The standard hypergeometric function ${}_2F_1(a, b; c; z)$ is defined as:

$$ {}_2F_1(a, b; c; z) := \sum_{n \geq 0} \frac{a^{\uparrow n} b^{\uparrow n}}{c^{\uparrow n}} \frac{z^n}{n!}, $$

where $x^{\uparrow n} := x(x+1)\cdots(x+n-1)$ is the n-th raising power of x.

An example: the star tree T_1

$$S_T = \sum_{a \geq 0} C_a^2 4^{-2a}$$

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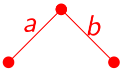
$$S_T = \sum_{a \geq 0} C_a^2 4^{-2a}$$
$$= 4 \,{}_2F_1(-\frac{1}{2}, -\frac{1}{2}; 1; 1) - 4.$$

An example: the star tree T_1

$$
\begin{aligned}
S_T &= \sum_{a\geq 0} C_a^2 4^{-2a} \\
&= 4\ {}_2F_1(-\frac{1}{2}, -\frac{1}{2}; 1; 1) - 4. \\
&= \frac{16}{\pi} - 4
\end{aligned}
$$
[Gauss, a long time ago.]

Another example and the Catalan identity



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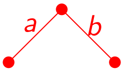

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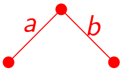

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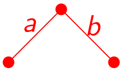

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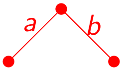
&= \sum_{x\geq 0} C_x C_{x+1} 4^{-2x} (\text{Catalan identity}).
\end{aligned}
$$

Another example and the Catalan identity



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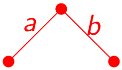

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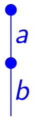

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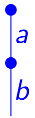

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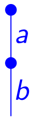

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so

$$2S_T = \sum_{a,x \geq 0} C_a C_x 4^{-a-x} + \sum_{a=x \geq 0} C_a C_x 4^{-a-x}$$

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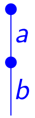

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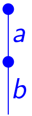

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- We needed the value of S_{T_2}!

Induction step: decorated trees

- For larger trees, we work by induction on the size of the tree.
- We need to consider more general trees, with *decorations* on the vertices.

And more...

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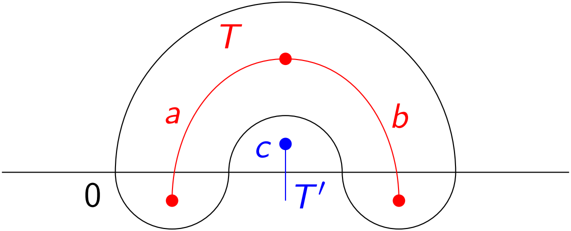
- Miracle: our method only works for trees and trees with one additional half-edge.
- Not yet implemented...needs to be done in a smart way.

THANKS



General graphs

- Our sums can be defined for general graphs G:

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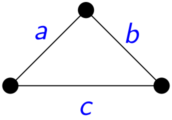
Question

In which vector space do the $\{S_G, G \text{ graph}\}$ live?

The triangle



$$S_\Delta = \sum_{a,b,c\geq 0} C_{a+b} C_{b+c} C_{a+c} 4^{-2a-2b-2c}$$

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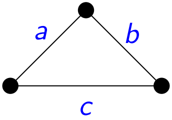

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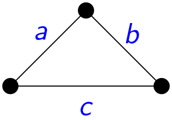
Theorem [Bostan, Féray & T. '25+]

We have

$$S_\Delta = 24 - 16\sqrt{2}.$$

The triangle



$$S_\Delta(t) = \sum_{a,b,c\geq 0} C_{a+b} C_{b+c} C_{a+c} t^{2a+2b+2c}$$

Theorem [Bostan, Féray & T. '25+]

We have

$$S_\Delta(t) = \frac{3}{2t^2} - 8\sqrt{2} - \frac{\sqrt{2}}{2t^2}\sqrt{1+\sqrt{1-16t^2}}.$$

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Question

In which vector space do the S_G, G graph live?

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Conjecture

The sum S_G is finite at $t = \frac{1}{16}$ if and only if, for any induced subgraph $H \subseteq G$, we have

$$\frac{3}{2}|V| > |E| + 1.$$

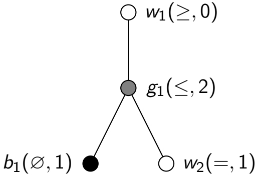
Decorated trees

Define a decorated tree T as follows.

- T is rooted;
- each vertex of T is either white, black or gray;
- each vertex v has a decoration $(\bowtie_v, K_v)$, where
  - $\bowtie_v \in \{\geq, \leq, =, \varnothing\}$;
  - $K_v \in \mathbb{Z}$;

We write $v_1 \leq v_2$ if v_2 is a descendent of v_1.

Example



Sums indexed by decorated trees

We associate to each white vertex w a variable ℓ_w, and to each black vertex b a black variable m_b.

We associate to each vertex of T a "restriction" R_v:

$$R_v : \sum_{\substack{w \leq v \\ w \in V_\circ(T)}} \ell_w \bowtie_v \sum_{\substack{b \leq v \\ b \in V_\bullet(T)}} m_b + \sum_{z \leq v} K_z.$$

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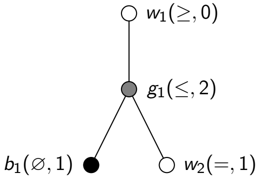
We define

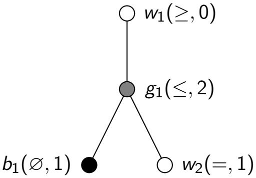
$$S_T = \sum_{(\ell_w)_{w \in V_\circ(T)}, (m_b)_{b \in V_\bullet(T)} \geq 0} \prod_w C_{\ell_w} 4^{-\ell_w} \prod_b C_{m_b} 4^{-m_b} \prod_{v \in V(T)} 1[R_v]$$

Example of a decorated tree



$$R_{w_1} : \ell_{w_1} + \ell_{w_2} \geq m_{b_1} + 4,$$
$$R_{g_1} : \ell_{w_1} \leq m_{b_1} + 4,$$
$$R_{b_1} : \varnothing,$$
$$R_{w_2} : \ell_{w_2} = 1.$$

Example of a decorated tree



$$R_{w_1} : \ell_{w_1} + \ell_{w_2} \geq m_{b_1} + 4,$$
$$R_{g_1} : \ell_{w_1} \leq m_{b_1} + 4,$$
$$R_{b_1} : \varnothing,$$
$$R_{w_2} : \ell_{w_2} = 1.$$

$$S_T = \sum_{\ell_{w_1}, \ell_{w_2}, m_{b_1}} C_{\ell_{w_1}} C_{\ell_{w_2}} C_{m_{b_1}} 4^{-\ell_{w_1} - \ell_{w_2} - m_{b_1}} 1[R_{w_1}, R_{g_1}, R_{w_2}].$$

From trees to decorated trees

A tree T

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- $S_T = \sum_{a,b} C_a C_b C_{a+b} 4^{-2a-2b}$

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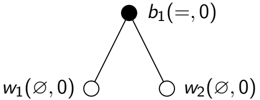

A decorated tree T'

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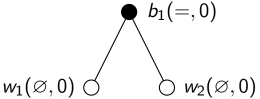

A decorated tree T'

- $S_T = \sum_{a,b} C_a C_b C_{a+b} 4^{-2a-2b}$
- $S_{T'} = \sum_{\ell_{w_1},\ell_{w_2},m_{b_1}\geq 0} C_{\ell_{w_1}} C_{\ell_{w_2}} C_{m_{b_1}} 4^{-\ell_{w_1}-\ell_{w_2}-m_{b_1}} 1[\ell_{w_1}+\ell_{w_2}-m_{b_1}=0].$
- $S_T = S_{T'}.$

From trees to decorated trees (2)

- Root a tree T at any of its vertices;
- Color in white (resp. black) the vertices at even (resp. odd) distance to the root.

Lemma Bostan, Féray & T. '25+

The map $(x_e) \in \mathbb{Z}_+^E \mapsto (X_v) \in \mathbb{Z}_+^V$ (where $X_v = \sum_{e \ni v} x_e$) is a bijection onto the set of V-tuples satisfying

$$(\star) \begin{cases} \sum_{w \in V_\circ(T)} X_w = \sum_{b \in V_\bullet(T)} X_b; \\ \forall w_0 \in V_\circ(T), \sum_{w \in V_\circ(T), w \leq w_0} X_w \geq \sum_{b \in V_\bullet(T), b \leq w_0} X_b; \\ \forall b_0 \in V_\bullet(T), \sum_{w \in V_\circ(T), w \leq b_0} X_w \leq \sum_{b \in V_\bullet(T), b \leq b_0} X_b. \end{cases}$$

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In particular, $S_T = \sum_{\substack{(X_v) \in \mathbb{Z}_+^V \\ (\star)}} C_{X_v} 4^{-X_v}$.

Sums on decorated trees

Theorem [Bostan, Féray & T. '25+]

For any decorated tree T, we have

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- Corollary: it holds for tree-indexed sums.

Some ideas for the proof

- Small decorated trees (height 1): expressed in terms of hypergeometric functions.

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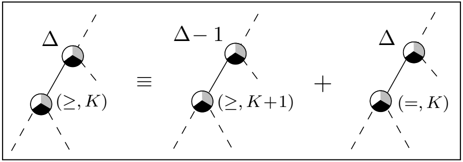
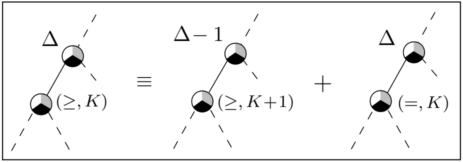
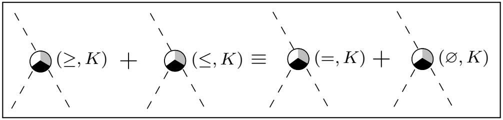
- Then, induction on the height of the tree.
- How to get from bigger trees to smaller ones?

Some tricks



Some tricks





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